THE

ELEMENTS

OF

ALGEBRA.

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PREFACE.

THE following pages have been compiled chiefly for the use of the Author's own Classes, and it has been his object to supply a useful and sufficiently comprehensive elementary work at as moderate a price as possible. With this view, the long and somewhat abstract demonstrations of the Rules and Principles employed in the Fundamental operations have been omitted, partly because they may be much better given orally, as circumstances require, and partly because they are totally unintelligible to a pupil of ordinary capacity until he has acquired some practical knowledge of the operations themselves, and a certain degree of facility in performing them. Most of these demonstrations, however, the learner is afterwards required to supply himself, in the form of Exercises, so as to cultivate his powers of analytical investigation, and train his mind to habits of correct thought. As the work progresses, and when the pupil may be supposed to be prepared for it, the Theoretic element is gradually introduced, and occupies the prominent place which it ought to do in every class-book.

The Exercises have been carefully prepared and arranged; some of them are selected from the best London and Cambridge works, and a considerable number from the Exami-

nation Papers for appointments to the Royal Artillery, and for admission to the Royal Military Academy at Woolwich. The greater part of these last have been placed at the end of Chapter XIII., and will be found to be by no means so difficult as many of the others in that chapter. A thorough acquaintance with what is here given will therefore form, so far as Algebra is concerned, an ample preparation for these important Examinations, in which, as is well known, Mathematical science occupies the first place.

ALGEBRA.

CHAPTER I.

DEFINITIONS.

1. In Algebra, the quantities under consideration are expressed by means of the letters of the alphabet. Known quantities are usually expressed by the first letters, as a, b, c, &c., and unknown quantities by the last, as x, y, z. Sometimes the letters of the Greek alphabet are employed, and sometimes also the capitals A, B, C, &c., according to circumstances.

2. The sign $+$ (*plus*) is the sign of Addition, and denotes that the quantity to which it is prefixed must be added. Such quantities are called *positive*.

3. The sign $-$ (*minus*) is the sign of Subtraction, and denotes that the quantity to which it is prefixed must be subtracted. Such quantities are called *negative*. Thus, if a represent 7, and b, 3; then $a + b$ represents 10, and $a - b$, 4.

Note. When no sign is prefixed, $+$ is always understood.

4. To denote the multiplication of quantities, the letters representing them are usually written in succession, as in a word. Thus ab denotes that the quantity represented by a is to be multiplied by the quantity represented by b. In some cases a point, or the sign $\times$ (*into*) is used to denote Multiplication. Thus $2.3.4$, or $2 \times 3 \times 4$, denotes the continued product of 2, 3, and 4.

5. If the same quantity be repeated as a factor any number of times, the product is called a *power* of that quantity.

When there are two factors, the product is called the *second power*, or *square;* when there are three, the *third power*, or *cube;* when there are four, the *fourth power;* and so on. Thus aa, aaa, $aaaa$, are respectively the second, third, and fourth powers of a. As, however, this notation would obviously be inconvenient, especially for the higher powers, it is usual to write the factor only once, placing above it, and a little to the right, a number to indicate how often it is repeated. Thus aa, aaa, $aaaa$, are written a^2, a^3, a^4; and the numbers 2, 3, and 4, are called the *indices* of those powers.

6. When one quantity is to be divided by another, the division is indicated by writing the dividend as numerator, and the divisor as denominator of a fraction; or writing the divisor after the dividend, with the sign $\div$ between them. Thus the expression $\frac{a}{b}$, or $a \div b$, denotes that a is to be divided by b.

7. The quantity a is called the second or square root of a^2; the third or cube root of a^3; the fourth root of a^4, and so on: and the signs $\sqrt{}$, $\sqrt[3]{}$, $\sqrt[4]{}$, &c., are used to denote the second, third, fourth, &c. roots respectively, of the quantities to which they are prefixed. These roots are also expressed by the fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c. used as indices. Thus the third root of a is written $\sqrt[3]{a}$, or $a^{\frac{1}{3}}$.

8. When a number is placed before a quantity as a multiplier, it is called the *coefficient* of that quantity. Thus in the expressions $3x$, ax; 3 and a are respectively the coefficients of x: the first a numeral one, and the second a literal one. (In fact, coefficient simply means co-factor; so that in the product ab, a is the coefficient of b, and b is the coefficient of a.)*

9. *Like* quantities are such as differ only in their coefficients: *Unlike* quantities consist of different combinations of letters. Thus $4ax$ and $6ax$ are like; but $4ax$ and $6a^2x$ are unlike. If $4a$ and $6a^2$ be considered the coefficients of x, then $4\,ax$ and $6a^2x$ are like.

10. A simple quantity consists of one term, as $4a$, $6a^2x$, &c. A compound quantity consists of more than one term, connected by the sign $+$ or $-$. Thus $a+b$, $a+bx-6b^2x$,

* When no number is prefixed the coefficient 1 is understood.

and $4a+2ab-3a^2x-4ab\,b^2x$, are compound quantities: the first is called a *binomial*, the second a *trinomial*, and the third a *polynomial*.

11. When any operation is to be performed on a compound quantity, that quantity is enclosed in brackets, as ( ), { }, [ ]. Thus $(a-b)\times x$ indicates that the quantity $a-b$ is to be multiplied by x.*

12. The sign $=$ (*equal to*), placed between two quantities, indicates that the former is equal to the latter.

13. The sign $>$ (*greater than*), between two quantities, signifies that the former is greater than the latter; and the sign $<$ (*less than*) that the former is less than the latter.

EXERCISES ON THE DEFINITIONS.

(1.) When $a=1$, $b=2$, $c=3$, what is the value of each of the following expressions? $a+b+c$; $-a+b+c$; $a-b+c$; $a+b-c$; $-a-b+c$; $-a+b-c$; $a-b-c$; $-a-b-c$.

Ans. 6; 4; 2; 0; 0; -2; -4; -6.

(2.) What do the same expressions become when $a=5$; $b=7$; $c=9$?

Ans. 21; 11; 7; 3; -3; -7; -11; -21.

(3.) If $a=4$, $b=3$, $c=7$, $x=5$, find the value of each of the following quantities: $3a+x$; $5a-4b+2x$; $12a+5b-3c-4x$; $3ac-4bc+cx$; $2abc-bcx$; $2abcx-acxx$.

Ans. 17; 18; 22; 35; 63; 140.

(4.) If $a=6$, $b=5$, $c=4$, $x=3$, what is the value of each of the following expressions? $5ac\div 2x$; $\frac{2a+x}{b}$; $\frac{3bc+2ax}{2cx}$; $\frac{2ab}{3c}-\frac{3ac+4bx}{2c+x}$; $\frac{2ab+cx}{ab-cx}-\frac{5ab-2cx}{ab+cx}$; $\frac{2ab-3ac+4ax}{3bc-2bx}-\frac{2bx+5cx}{2ax-3bx+2cx}$.

Ans. 20; 3; 4; -7; 1; -4.

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* Sometimes a line is drawn over the quantity instead of a vinculum. Thus $\overline{a-b}\times x$ is the same as $(a-b)\times x$.

(5.) If $a=1$, $b=2$, $c=8$, $x=4$, find the value of each of the following quantities: $a^2-b^2+c^2-x^2$; $4a^2b-7b^2c+c^2x$; $a^2b^3-2b^2c^3+c^2x^3$; $ab^2c^3-2a^2bx^3+a^3c^2x-2b^3cx^2$. Ans. 45; 40; 8; 0.

(6.) Find the value of each of the following quantities on the same supposition: $\sqrt{2c}-\sqrt{x}$; $\sqrt[3]{bx}+\sqrt[3]{ac}$; $(a+c)^{\frac{1}{2}}$; $(a^2-2ab+b^2)^{\frac{1}{2}}$; $(a^2+2ab+b^2)^{\frac{1}{2}}$; $(a^3+3a^2b+3ab^2+b^3)^{\frac{1}{3}}$. Ans. 2; 4; 3; 1; 3; 3.

(7.) If $x=8$, $a=6$, what is the difference between x^2 and $2x$; x^3 and $3x$; x^4 and $4x$; $x+a^3$ and $(x+a)^3$; $4x+a$ and $4(x+a)$; x^2+a^2 and $(x+a)^2$; $\sqrt{x^2}+\sqrt{a^2}$ and $\sqrt{(x^2+a^2)}$; $x+a$ and $\sqrt{(x^2+a^2)}$?

Ans. 48; 488; 4064; 2520; 18; 96; 4; 4.

(8.) If $a=3$, $b=4$, $c=5$, $s=\frac{1}{2}(a+b+c)$, what is the value of each of the following quantities? $\sqrt{s(s-a)(s-b)(s-c)}$; $\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$; $\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$; $\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$.

Ans. 6; $\frac{1}{3}$; $\frac{1}{2}$; 1.

CHAPTER II.

ADDITION.

14. If the quantities be like, and have like signs; add the coefficients, to the sum prefix the common sign, and annex the common letter or letters.

15. If the quantities be like, and have unlike signs; add the $+$ and $-$ coefficients separately, and take the difference of the sums, to which prefix the sign of the greater, and annex the common letter or letters.

16. If the quantities be unlike, connect them together with their proper signs.

EXERCISES.

Add together,

(1.) $2a$, and $3a$. . . . . Ans. $5a$.
(2.) $4a$, and $7a$. . . . . . $11a$.
(3.) $5a$, and $2a$. . . . . . . $7a$.
(4.) $5a$, and $-2a$. . . . . . $3a$.
(5.) $3a$, and $-a$. . . . . . $2a$.
(6.) $-2a$, and $-3a$. . . . . $-5a$.
(7.) $-4a$, and $-7a$. . . . . $-11a$.
(8.) a, and b. . . . . . . $a+b$.
(9.) a, and x. . . . . . $a+x$.
(10.) b, and $-x$. . . . . . $b-x$.
(11.) $a+b$, and $a+b$. . . . . $2a+2b$.
(12.) $a-b$, and $a-b$. . . . . $2a-2b$.
(13.) $a-b+c$; and $a+b-c$. . . $2a$.
(14.) $a+b-c$; $a-b+c$; and $-a+b+c$. $a+b+c$.
(15.) $4a+2x-2y$; $3a+2x-y$; $2a+4x-3y$. $9a+8x-6y$.
(16.) $4x-3y$; $2x+y$; $x+3y$; & $-5x+2y$. $2x+3y$.
(17.) $4-b$; $-3-2b$; $-4+6b$; and $3-b$. $2b$.
(18.) $3a+4b+x$; $2a-3b+3x$; & $-2a-2b+3x$. $3a-b+7x$.
(19.) $a^2+2ax+x^2$; and $a^2-2ax+x^2$. $2a^2+2x^2$.

(20.) $4x^3-2x^2+4x-1$; $-2x^3+4x^2-x+4$;
and $4x^3-x^2+4x-2$. . $6x^3+x^2+7x+1$.

(21.) $x^3+2a^2x-3ax^2$; $-a^2x+2ax^2$; and
x^3+a^3. . . . . $2x^3-ax^2+a^2x+a^3$.

(22.) $ac+bd$; $bd-cd$; $ac+cd$. . . $2ac.+2bd$.

(23.) $2x^2-3ax+2a^2$; $3x^2+2ax-4a^2$;
$x^2+ax-2a^2$; x^2+a^2. . . $7x^2-3a^2$.

(24.) $ab-2ad+3bd$; $-ac+2bc-3cd$;
$ad-2bd+3ab$; $-bc+2cd-3ac$;
$bd-2ab+3ad$; $-cd+2ac-3bc$.
$2ab-2ac+2ad-2bc+2bd-2cd$.

SUBTRACTION.

17. RULE. Change the signs of the quantity to be subtracted, or rather, suppose them to be changed, and proceed as in Addition.

EXERCISES.

(1.) From $6a$, take $2a$. . . . Ans. $4a$.
(2.) ... $6a$, take $-2a$. . . . $8a$.
(3.) ... 1, take $1-a$. . . . . a.
(4.) ... a, take b. . . . : $a-b$.
(5.) ... a, take $-b$. . . . . $a+b$.
(6.) ... a, take $b+x$, . . . $a-b-x$.
(7.) ... $a+b$, take $a-b$. . . . $2b$.
(8.) ... 1, take $1+a$. . . . $-a$.
(9.) ... $a+b-c$, take $a-b+c$. . . $2b-2c$.
(10.) ... $4a-2b-3c$, take $2a-3b+2c$. $2a+b-5c$.
(11.) ... $3a+2x-y$, take $a-2x+3y$. $2a+4x-4y$.
(12.) ... $2ax+by-c$, take $2ax-by+c$. $2by-2c$.
(13.) ... $6-a^2$, take a^2-6. . . $12-2a^2$.
(14.) ... $2ab-3ac+2bc$, take $2ab+3ac-2bc$. $-6ac+4bc$.
(15.) ... $4a^2+7b^2-5x^2$, take a^2-b^2+x. $3a^2+8b^2-6x^2$.
(16.) ... $a^2+2ab+b^2$, take $a^2-2ab+b^2$. $4ab$.
(17.) ... $2x^3+4x^2-3x-1$, take $2x^3+2x^2-3x+1$. . . . . $2x^2-2$.
(18.) ... $3a-4b+5c-6x+7y$, take $-a+b-c+x-y$. . $4a-5b+6c-7x+8y$.
(19.) ... $x^3+3x^2y+3xy^2+y^3$, take $x^3-3x^2y+3xy^2-y^3$. . . $6x^2y+2y^3$.
(20.) ... $ab-2ac+3ad-4bc+5bd-6cd$, take $ab-2ac+ad-2bc+5bd-3cd$. . . . . $2ad-2bc-3cd$.

MULTIPLICATION.

18. When both factors are simple quantities. Find the product of the numerical coefficients, to which annex the product of the letters (see Def. 4). If the factors have both the same sign, the product is *positive*; if they have different signs, it is *negative*.

NOTE. Powers of the same quantity are multiplied together by adding the indices.

19. When one factor is compound. Multiply each of its terms by the other factor.

20. When both factors are compound. Multiply each term of the multiplicand by each term of the multiplier, and add the partial products, as in Arithmetic.

EXERCISES.

Multiply

(1.) a by b. . . . . . . Ans. ab.
(2.) ax by y. . . . . . . axy.
(3.) ab by xy. . . . . $abxy$.
(4.) a^2 by a^3. . . . . . . a^5.
(5.) $3a$ by $-b$. . . . . . . $-3ab$.
(6.) $-4ax$ by $3a$. . . . . . $-12a^2x$.
(7.) $-3a^2b$ by $-2ab^2$. . . . . . $6a^3b^3$.
(8.) $-2ab^2c$ by $-a^2xy^2$. . . . . $2a^3b^2cxy^2$.
(9.) $\sqrt{a}$ by $\sqrt{a}$. . . . . . . a.
(10.) $3\sqrt{5}$ by $2\sqrt{5}$. . . . . . 30.
(11.) a^m by a^n. . . . . a^{m+n}.
(12.) a^{m+1} by a^{m-1}. . . . . . a^{2m}.
(13.) $a+b$ by a. . . . . . . a^2+ab.
(14.) $a-b+x$ by 3. . . . . . $3a-3b+3x$.
(15.) $ax-bx^2$ by c. . . . . . . $acx-bcx^2$.
(16.) $1-2x+3x^2-4x^3$ by $2ax$. $2ax-4ax^2+6ax^3-8ax^4$.
(17.) $2ab-3ac-2bc$ by $-3abc$. $-6a^2b^2c+9a^2bc^2+6ab^2c^2$.
(18.) $1-2ax+3bx^2$ by $-2abx$. $-2abx+4a^2bx^2-6ab^2x^3$.
(19.) $a+b$ by $a+b$. . . . $a^2+2ab+b^2$.
(20.) $a+b$ by $a-b$. . . . a^2-b^2.
(21.) $a-b$ by $a-b$. . . $a^2-2ab+b^2$.
(22.) $a+2x$ by $a+2x$. . . $a^2+4ax+4x^2$.

(23.) $2a-3x$ by $2a-3x$. Ans. $4a^2-12ax+9x^2$.

(24.) $3a-x$ by $2a-y$. . . . $6a^2-2ax-3ay+xy$.

(25.) $2a+b$ by $x+2y$. . . . $2ax+bx+4ay+2by$.

(26.) $3a^2+2x^2$ by $3a^2+2x^2$. . $9a^4+12a^2x^2+4x^4$.

(27.) $2a+3$ by $3a+2$. . . . $6a^2+13a+6$.

(28.) $4x-7$ by $4x-3$. . . $16x^2-40x+21$.

(29.) $a^2-2ax+x^2$ by $a-x$. . $a^3-3a^2x+3ax^2-x^3$.

(30.) a^2+ax+x^2 by $a-x$. . a^3-x^3.

(31.) a^4+a^2+1 by a^2-1. . a^6-1.

(32.) x^2+2x+4 by x^2-2x+4. . x^4+4x^2+16.

(33.) $1+2x+3x^2+4x^3+5x^4$
by $1-2x+x^2$ . . . . $1-6x^5+5x^6$.

(34.) $1+\sqrt{2}$ by $1-\sqrt{2}$. . . -1.

(35.) a^n+x^n by $a-x$. . . . $a^{n+1}+ax^n-a^nx-x^{n+1}$.

(36.) $a^n-2a^{n-1}+3a^{n-2}-4a^{n-3}$ by $a+1$.
$a^{n+1}-a^n+a^{n-1}-a^{n-2}-4a^{n-3}$.

(37.) $(a-1)(a+3)(a+5)(a-7)$.
$a^4-42a^2-64a+105$.

(38.) $(a-x)(a+3x)(a+5x)(a-7x)$.
$a^4-42a^2x^2-64ax^3+105x^4$.

(39.) $(2+\sqrt{2})(2-\sqrt{2})(3+\sqrt{5})(3-\sqrt{5})$. . . 8.

(40.) $(a+b+c)(a+b-c)(a-b+c)(-a+b+c)$.
$2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4$.

(41.) $(a+x)^4$. . $a^4+4a^3x+6a^2x^2+4ax^3+x^4$.

(42.) $(a-x)^5$. . $a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$.

DIVISION.

21. When both divisor and dividend are simple quantities: Place the dividend over the divisor so as to form a fraction (Def. 6), removing any factor common to both numerator and denominator; and if the divisor and dividend have the same sign the quotient is *positive;* if they have different signs it is *negative.*

If the divisor is an exact factor of the dividend, the

quotient may be found by rejecting that factor from the dividend.

A power of a quantity is divided by any other power of the same quantity, by subtracting the index of the divisor from that of the dividend.

22. When the dividend is a compound quantity and the divisor a simple one; divide each term of the dividend by the divisor.

23. When the divisor is a compound quantity; arrange, if possible, the terms of the divisor and dividend according to the powers of some letter common to both, beginning with the highest power and going regularly down to the lowest, or *vice versa*; then proceed as in long Division in Arithmetic.

EXERCISES.

Divide

(1.) ax by a. . . . . . . Ans. x.
(2.) ax by x. . . . . a.
(3.) $5b$ by 5. . . . . b.
(4.) $3ab$ by $3b$. . . . . a.
(5.) $8abx$ by $2bx$. . . . . $4a$.
(6.) $4xy$ by $-2y$. . . . . . . $-2x$.
(7.) a^4 by $2a$. . . . . $\frac{1}{2}a^3$.
(8.) $6a^2b^3$ by $-4ab^2c$. . . . . . $-\frac{3}{2}\frac{ab}{c}$.
(9.) $-x$ by -1. . . . . x.
(10.) -1 by -1. . . . . . . 1.
(11.) a by $\sqrt{a}$. . . . . $\sqrt{a}$.
(12.) a^{2n} by a^n. . . . . . . a^n.
(13.) $-a^m$ by $-a^n$. . . . . . . a^{m-n}.
(14.) $-a^{m+1}$ by a^{m-1}. . . . . . $-a^2$.
(15.) $-10a^2b^nx^3$ by $2ax$. . . . . . $-5ab^nx^2$.
(16.) $6a^nb^2c^3$ by $-3abc$. . . . . $-2a^{n-1}bc^2$.
(17.) a^3 by a^3, and prove $a^0=1$. $a^3\div a^3=a^0$ & $\frac{a^3}{a^3}=\frac{1}{1}=1$.
(18.) x^3 by x^5, and prove $x^{-2}=\frac{1}{x^2}$. $x^3\div x^5=x^{-2}$ & $\frac{x^3}{x^5}=\frac{1}{x^2}$.
(19.) $3ab-2ac$ by a. . . . . $3b-2c$.
(20.) $4a^2bx-6a^3bx^2$ by $2a^2x$. . . $2b-3abx$.
(21.) $3ab-12a^2bc+9ab^2c^2$ by $3ab$. . . $1-4ac+3bc^2$.

(22.) a^2-2a+3 by -1. . Ans. $-a^2+2a-3$.
(23.) $2x^3-4ax^2+6a^2x$ by $-2x$. . $-x^2+2ax-3a^2$.
(24.) $3a^2b^3x-4ab^2x^3+5a^3bx^2$ by $6abx$. $\frac{1}{2}ab^2-\frac{2}{3}bx^2+\frac{5}{6}a^2x$.
(25.) $a^2+2ab+b^2$ by $a+b$. . . $a+b$.
(26.) $a^2-2ab+b^2$ by $a-b$. . $a-b$.
(27.) a^2-b^2 by $a-b$. . . . $a+b$.
(28.) a^2-b^2 by $a+b$. . . $a-b$.
(29.) x^2+x-6 by $x-2$. . . . $x+3$.
(30.) x^2+x-6 by $x+3$. . . $x-2$.
(31.) $4a^2-15x^2-4ax$ by $2a+3x$. . $2a-5x$.
(32.) $4a^4-64$ by $2a-4$. . $2a^3+4a^2+8a+16$.
(33.) a^4-x^4 by $a-x$. . . $a^3+a^2x+ax^2+x^3$.
(34.) a^4-x^4 by $a+x$. . . $a^3-a^2x+ax^2-x^3$.
(35.) a^5-x^5 by $a-x$. . . $a^4+a^3x+a^2x^2+ax^3+x^4$.
(36.) a^5-x^5 by $a+x$. . . $a^4-a^3x+a^2x^2-ax^3+x^4$.
(37.) a^4+x^4 by $a-x$. . . $a^3+a^2x+ax^2+x^3+\frac{2x^4}{a-x}$.
(38.) a^4+x^4 by $a+x$. . . $a^3-a^2x+ax^2-x^3+\frac{2x^4}{a+x}$.
(39.) a^5+x^5 by $a-x$. $a^4+a^3x+a^2x^2+ax^3+x^4+\frac{2x^5}{a-x}$.
(40.) a^5+x^5 by $a+x$. . . $a^4-a^3x+a^2x^2-ax^3+x^4$.
(41.) $2x^4-x^3+x^2+7x+3$ by x^2-2x+3. $2x^2+3x+1$.
(42.) $3a^2-4b^2+3c^2-4ab+6ac-4bc$
by $a-2b+c$. . . . . $3a+2b+3c$.
(43.) $12a^4-a^3x-21a^2x^2+22ax^3-6x^4$
by $3a^2-4ax+2x^2$. . . . $4a^2+5ax-3x^2$.
(44.) $a^2x^3+a^5-2abx^3+b^2x^3+a^3b^2-2a^4b$,
by $ax-bx+a^2-ab$. $ax^2-bx^2-a^2x+abx+a^3-a^2b$.
(45.) a by $a-1$.

$$1+\frac{1}{a}+\frac{1}{a^2}+\frac{1}{a^3}+\text{\&c.}=1+a^{-1}+a^{-2}+a^{-3}+\text{\&c.}$$

(46.) a by $a+1$.

$$1-\frac{1}{a}+\frac{1}{a^2}-\frac{1}{a^3}+\text{\&c.}=1-a^{-1}+a^{-2}-a^{-3}+\text{\&c.}$$

(47) a by a^2-1. $\frac{1}{a}+\frac{1}{a^3}+\frac{1}{a^5}+\text{\&c.}=a^{-1}+a^{-3}+a^{-5}+\text{\&c.}$

(48.) a by $a+b$.

$$1-\frac{b}{a}+\frac{b^2}{a^2}-\frac{b^3}{a^3}+\&c.=1-a^{-1}b+a^{-2}b^2-a^{-3}b^3+\&c.$$

(49.) a by $a-b$.

$$1+\frac{b}{a}+\frac{b^2}{a^2}+\&c.=1+a^{-1}b+a^{-2}b^2+\&c.$$

(50.) a by a^2-2b^2.

$$\frac{1}{a}+\frac{2b^2}{a^3}+\frac{4b^4}{a^5}+\&c.=\frac{1}{2b}(2a^{-1}b+4a^{-3}b^3+8a^{-5}b^5+\&c.)$$

MULTIPLICATION BY DETACHED COEFFICIENTS.

24. When the factors to be multiplied together consist of powers of the same quantity, either in ascending or descending order, it is obvious that in the product the powers of the same quantity will ascend or descend in the same order. Hence the operations may be performed by means of the coefficients alone, annexing afterwards the proper powers to the coefficients so obtained. Thus to multiply x^3-2x^2+3x-1 by $2x^2+3x+2$.

Write out the coefficients of the multiplier and multiplicand in their proper order, and with their proper signs, and proceed in precisely the same manner as if they were connected with the several letters. Then the literal part of the first term being $x^3\times x^2=x^5$, we insert x^5 after the coefficient 2, and the successive powers after the others, till we come to the last, -2, to which (being the product of the terms not containing x) no power is annexed.

$$\begin{array}{rrrrrr}
1 & -2 & +3 & -1 & & \\
2 & +3 & +2 & & & \\
\hline
2 & -4 & +6 & -2 & & \\
 & 3 & -6 & +9 & -3 & \\
 & & 2 & -4 & +6 & -2 \\
\hline
2 & -1 & +2 & +3 & +3 & -2
\end{array}$$

Ans. $2x^5-x^4+2x^3+3x^2+3x-2$

If any term is wanting to complete the regular succession of powers in either of the factors, its place must be supplied with a cipher.

Multiply

EXERCISES.

(1.) $2a-3$ by $2a-3$. . . Ans. $4a^2-12a+9$.

(2.) $4x^2-6x+9$ by $2x+3$. . . $8x^3+27$.

(3.) $1-2a+3a^2-4a^3$ by $1+a$.

Ans. $1-a+a^2-a^3-4a^4$.

(4.) $a^3-3a^2x+3ax^2-x^3$ by $a-x$.

$a^4-4a^3x+6a^2x^2-4ax^3+x^4$.

(5.) $a^3-a^2x+ax^2-x^3$ by $a+x$. . . . a^4-x^4.

(6.) $4x^2+12xy+9y^2$ by $4x^2-12xy+9y^2$.

$16x^4-72x^2y^2+81y^4$.

(7.) $x^2+3ax+4a^2$ by $x^2-3ax+4a^2$. $x^4-a^2x^2+16a^4$.

(8.) $a^4+a^2x^2+x^4$ by a^2-x^2. . . . a^6-x^6.

(9.) $(x+1)(x-2)(x-3)(x+4)$. $x^4-15x^2+10x+24$.

(10.) $(x-3)(x+4)(x-5)(x+6)$.

$x^4+2x^3-41x^2-42x+360$.

(11.) $(a+2x)(2a-x)(2a-3x)(3a+2x)$.

$12a^4+8a^3x-39a^2x^2-8ax^3+12x^4$.

(12.) $(a-2x)^5$.

$a^5-10a^4x+40a^3x^2-80a^2x^3+80ax^4-32x^5$.

DIVISION BY DETACHED COEFFICIENTS.

25. The same method may also be employed with great advantage in many cases in Division, wherever the divisor and dividend are such that the literal part of the terms of the quotient can be supplied by inspection. This method will be understood from the following examples:—Divide $3x^4+14x^3+12x+9$ by x^2+5x+1.

$$\begin{array}{l}
3+14+0+12+9\,\big(\,1+5+1 \\
\underline{3+15+3} \qquad\qquad\; 3-1+2 \\
\quad -1-3+12 \quad \text{Ans. } 3x^2-x+2+\dfrac{3x+7}{x^2+5x+1} \\
\quad \underline{-1-5-1} \\
\qquad\quad 2+13+9 \\
\qquad\quad \underline{2+10+2} \\
\qquad\qquad\quad 3+7
\end{array}$$

Here placing the coefficients as in the margin, those of the divisor being to the right hand, immediately above those of the quotient, we find in the usual way, 3 for the first figure of the quotient; then multiplying the terms of the divisor by 3, the products are placed respectively under the three first terms of the dividend, and

the work proceeds exactly as if the letters were annexed; and since $\frac{x^4}{x^2}=x^2$, the literal part of the first term of the quotient is x^2, and the powers will descend in the same order as in the divisor and dividend, the quotient therefore is $3x^2-x+2+\frac{3x+7}{x^2+5x+1}$.

The process may be still farther abridged, as follows:—

$$\begin{array}{l}3+14+0+12+9\ \big(\ 1+5+1\\ -3-15-3 \qquad\qquad\ \ 3-1+2\\ \qquad 1+5+\ 1\\ \qquad\ -2-10-2\\ \hline \qquad\qquad 3+7\end{array}$$

Arrange the coefficients as before, and find the first figure of the quotient, which is 3; multiply the terms of the divisor by 3, and set down the products under the dividend as before, but with their signs changed; then $14-15=-1$, which divided by 1, the first term of the divisor, gives -1 for the next figure of the quotient; multiply, and set down the products as before, with their signs changed; then $0-3+5=2$, which again divided by 1, the first term of the divisor, gives 2 for the next figure of the quotient; multiply as before, and set down the products with their signs changed. Then supplying the letters, the quotient is $3x^2-x+2$ as before, with the remainder $\frac{3x+7}{x^2+5x+1}$.

A little consideration will show that the two processes are identical, the several subtractions in the first being omitted in the second, and the signs of the partial products being conceived to be changed in the first, while they are actually changed in the second.

When the divisor consists of two terms, and the coefficient of the first is unity, the division may be performed as follows:

To divide $4x^4-33x^2+8x+3$ by $x+3$. Arrange the coefficients as in the margin, omitting the first term of the divisor; then multiply the first term 4 of the dividend by the 3 of the divisor; set down the product 12 under the second term, and subtract; the remainder is—12; which write down below; multiply the -12 by the 3 of the divisor, set down the product -36 under the next term, and

$$\begin{array}{l}4+0-33+8+3\ (\ +3\\ \underline{\quad 12-36+9-3}\\ \quad -12+\ 3-1\ (+6)\end{array}$$

Ans. $4x^3-12x^2+3x-1+\frac{6}{x+3}$

subtract; the remainder is $+3$, which again write down below; proceed in like manner to the end, multiplying each remainder by the 3 of the divisor, and subtracting the product from the next term. In this case, the coefficient of the first term of the divisor being 1, to divide by it produces no alteration, and hence the first coefficient of the dividend, with the several remainders except the last, may be taken for the coefficients of the quotient.

Divide

EXERCISES.

(1.) $4x^2-4x-15$ by $2x+3$. . . . Ans. $2x-5$.

(2.) $a^2-2ab+b^2$ by $a-b$. . . . . . $a-b$.

(3.) $2x^4-x^3+x^2+7x+3$ by x^2-2x+3. $2x^2+3x+1$.

(4.) $12x^4-x^3-4x^2+7x-2$ by $3x^2+2x-1$. $4x^2-3x+2$.

(5.) $x^4+4x^3+6x^2+4x+1$ by $x+1$. x^3+3x^2+3x+1.

(6.) $3x^4-10x^3+11x^2-8x+6$ by $x-2$.

$3x^3-4x^2+3x-2+\frac{2}{x-2}$.

(7.) $3x^4-4ax^3-12a^2x^2-7a^3x+a^4$ by $x-3a$.

$3x^3+5ax^2+3a^2x+2a^3+\frac{7a^4}{x-3a}$.

(8.) $4a^5+14a^4x-5a^3x^2+8a^2x^3-14ax^4+8x^5$ by $a+4x$.

$4a^4-2a^3x+3a^2x^2-4ax^3+2x^4$.

(9.) $12a^4-a^3x-21a^2x^2+22ax^3-6x^4$ by $3a^2-4ax+2x^2$.

$4a^2+5ax-3x^2$.

(10.) $6x^5-x^4y+7x^3y^2+3x^2y^3+8xy^4+6y^5$ by $2x^2-3xy+3y^2$. $3x^3+4x^2y+5xy^2+3y^3+\frac{2xy^4-3y^5}{2x^2-3xy+3y^2}$.

(11.) $32a^5+243x^5$ by $2a+3x$.

$16a^4-24a^3x+36a^2x^2-54ax^3+81x^4$.

(12.) a^5-32x^5 by $a-2x$. $a^4+2a^3x+4a^2x^2+8ax^3+16x^4$.

(13.) $1+x^2+x^4$ by $1+x+x^2$. . . . . . $1-x+x^2$.

(14.) $15a^4+10a^3x+4a^2x^2+6ax^3-3x^4$ by $3a^2-x^2+2ax$.

$5a^2+3x^2$.

MISCELLANEOUS EXERCISES.

1. Express algebraically the sum and difference of two numbers; then, *first*, add the two expressions; *second*, subtract the less from the greater; and, *third*, multiply the one

by the other: what are the three results expressed in English?

Ans. Twice the greater; twice the less; and the difference of the squares.

2. The sum of two numbers is 20, and their difference 12; find the numbers. Ans. 16 and 4.

3. Half the sum of two numbers is 15, and half their difference 8; find the numbers. Ans. 23 and 7.

4. If A can do a piece of work in 12 days, express the parts done in 1 day, and also in x days. Ans. $\frac{1}{12}$ and $\frac{x}{12}$.

5. If x be the less of two numbers, and 8 the difference, express the greater. Ans. $x+8$.

6. If x be the greater of two numbers, and 8 the difference, express the less. Ans. $x-8$.

7. Express the square of the sum of two quantities, a and b, and also the sum of their squares; and find the difference.

Ans. $a^2+2ab+b^2$; a^2+b^2; $2ab$.

8. If x represent a number of shillings, express the same sum of money in half-crowns. Ans. $\frac{2x}{5}$ half-crowns.

9. One of two brothers is as much above 25 years of age as the other is below it; if $25+x$ is the age of the one, what is the age of the other? Ans. $25-x$.

10. If x articles cost y pounds, find the price of each. Ans. $\frac{y}{x}$.

11. If one of two numbers in the ratio of m to n be expressed by mx, find the expression for the other. Ans. nx.

12. A sum of money is to be divided among three persons, A, B, and C; if x represent A's share, find B's and C's, when B's is double, and C's treble of A's; also when B's is half, and C's the third part of A's. Ans. $2x$ and $3x$; $\frac{x}{2}$ and $\frac{x}{3}$.

13. If A can do a piece of work in a days, and B in b days, find the part done by each in x days. Ans. $\frac{x}{a}$ and $\frac{x}{b}$.

14. Express the two numbers whose digits are x and y; and x, y, and z. Ans. $10x+y$; and $100x+10y+z$.

15. What does x^2+y^2 want of being a complete square?

Ans. $\pm 2xy$.

CHAPTER III.

FRACTIONS.

FUNDAMENTAL PRINCIPLES.

26. If the numerator and denominator of a fraction be both multiplied by the same number, or both divided by the same number, the value of the fraction is not changed.

27. A fraction is multiplied by any quantity, either by multiplying the numerator or dividing the denominator by that quantity.

28. A fraction is divided by any quantity, either by dividing the numerator, or multiplying the denominator by that quantity.

29. It follows from the first of these principles that a fraction may be reduced to its lowest terms by dividing its numerator and denominator by any factors which are common to both. In many cases such factors are easily discovered by inspection, and their product is called the *Greatest Common Measure* of the two quantities.

30. To find the Greatest Common Measure of any two quantities:—

First, When it is a simple quantity. To the greatest common measure of the numerical coefficients annex the common letters.

Second, When it is a compound quantity. (1.) Divide the one given quantity by the other, using as divisor the one which is of lower dimensions, if the dimensions be different. (2.) If there be a remainder, divide the divisor by it. (3.) If there still be a remainder, divide the last divisor by it; and continue the process till nothing remain. The last divisor is the Greatest Common Measure.

This rule depends on the two following principles:—

1. If one quantity measure another, it will measure any multiple of it.

2. If one quantity measure two others, it will measure their sum or difference.

In order to simplify the operation, either of the original quantities, or any of the remainders, must be divided by any

simple quantity common to all the terms; and, to avoid fractions in the quotients, any dividend may be multiplied by any simple quantity. Care must be taken, however, that no factor rejected be common to both of the original quantities. If there be such a common factor, it is better to divide both quantities by it, and afterwards to introduce it as a factor into the common measure.

The Greatest Common Measure of three or more quantities may be found, by first finding the Greatest Common Measure of any two of them, then of that and a third, and so on.

The following example will show the method of applying the above rule. To find the Greatest Common Measure of $8x^3+6x^2-4x-3$, and $12x^3+5x^2+x+3$.

Writing out the coefficients as in ordinary Division, since the quantities are of the same dimension, we may divide either by the other. Doubling the latter to make the first term divisible by 8, and dividing, we get 3 for quotient, and the remainder is $-8+14+15$; then dividing the divisor by this, we first get for quotient -1, with the remainder $20+11-3$, and doubling this to make the first term divisible by -8, we get for the next part of the quotient -5, with the remainder $92+69$, or by division by 23, $4+3$. Lastly, Dividing the last divisor by this remainder, we get $-2+5$ for quotient, and no remainder; hence $4+3$ are the coefficients of the Greatest Common Measure, which is therefore $4x+3$.*

$8+6-4-3$	$12+5+1+3$	
$8-14-15$	2	
$20+11-3$	$24+10+2+6$	3
2	$24+18-12-9$	
$40+22-6$	$-8+14+15$	$-1-5$
$40-70-75$	$-8-6$	
$23)\ 92+69$	$20+15$	
$4+3$	$20+15$	$-2+5$
or $4x+3$		

It is usual in operations such as the above to write the dividend always on the right-hand side of the divisor; but this serves no good purpose, as the work is just as easy when the position of each is reversed.

* Since the first sign in the divisor, $-8+14+15$, is $-$, all the signs might have been changed, or it might have been multiplied by -1.

EXERCISES.

Find the Greatest Common Measure of the terms of the following fractions, and reduce them to their lowest terms:—

(1.)	$\frac{125}{900}$.	Ans. 25	$\frac{5}{36}$
(2.)	$\frac{ax}{bx}$.	x	$\frac{a}{b}$
(3.)	$\frac{5a^3bx^2}{15a^2bx^3}$.	$5a^2bx^2$	$\frac{a}{3x}$
(4.)	$\frac{4x^5y^3}{6x^3y^5}$.	$2x^3y^3$	$\frac{2x^2}{3y^2}$
(5.)	$\frac{x^2-y^2}{(x+y)^2}$.	$x+y$	$\frac{x-y}{x+y}$
(6.)	$\frac{ab+bx}{ay+xy}$.	$a+x$	$\frac{b}{y}$
(7.)	$\frac{x^3+ax^2}{xy^2+ay^2}$.	$x+a$	$\frac{x^2}{y^2}$
(8.)	$\frac{a^3-ab^2}{ab^2+b^3}$.	$a+b$	$\frac{a^2-ab}{b^2}$
(9.)	$\frac{3ax-3x^3}{5ax-5x^3}$.	$ax-x^3$	$\frac{3}{5}$
(10.)	$\frac{6a^2-2ax}{9ac-3cx}$.	$3a-x$	$\frac{2a}{3c}$
(11.)	$\frac{(x-y)^4}{(x^2-y^2)^2}$.	$(x-y)^2$	$\frac{(x-y)^2}{(x+y)^2}$
(12.)	$\frac{x^3+y^3}{(x+y)^2}$.	$x+y$	$\frac{x^2-xy+y^2}{x+y}$
(13.)	$\frac{9a^2-4x^2}{27a^3-8x^3}$.	$3a-2x$	$\frac{3a+2x}{9a^2+6ax+4x^2}$
(14.)	$\frac{3x^3-2x^2-2x+3}{2x^3+3x^2+3x+2}$.	$x+1$	$\frac{3x^2-5x+3}{2x^2+x+2}$
(15.)	$\frac{2x^3-x^2+x-2}{3x^3-5x^2+5x-3}$.	$x-1$	$\frac{2x^2+x+2}{3x^2-2x+3}$
(16.)	$\frac{4+12x+9x^2}{2+13x+15x^2}$.	$2+3x$	$\frac{2+3x}{1+5x}$

(17.) $\dfrac{2-7x+6x^2}{4-4x-3x^2}$. . . . Ans. $2-3x$. $\dfrac{1-2x}{2+x}$

(18.) $\dfrac{2x^3+9x^2-2x+15}{2x^3+13x^2-4x+21}$. . . . $2x^2-x+3$ $\dfrac{x+5}{x+7}$

(19.) $\dfrac{x^4-25x^2+40x-16}{x^4+2x^3-18x^2+17x-4}$. . x^2+5x-4 $\dfrac{x^2-5x+4}{x^2-3x+1}$

(20.) $\dfrac{x^3-11x^2+39x-45}{3x^2-22x+39}$. . . $x-3$. $\dfrac{x^2-8x+15}{3x-13}$

(21.) $\dfrac{3x^3-13x^2+23x-21}{6x^3+x^2-44x+21}$. . . . $3x-7$. $\dfrac{x^2-2x+3}{2x^2+5x-3}$

(22.) $\dfrac{x^4-41x^2+16}{x^4-7x^3+28x-16}$. . . . x^2-7x+4 $\dfrac{x^2+7x+4}{x^2-4}$

(23.) $\dfrac{8x^4-4x^2+27x-4}{2x^4+9x^3-30x^2+48x-32}$. $2x^2-3x+4$ $\dfrac{4x^2+6x-1}{x^2+6x-8}$

(24.) $\dfrac{3a^4-7a^3x+8a^2x^2+ax^3-x^4}{9a^4+3a^3x-8a^2x^2+5ax^3-x^4}$. $3a-x$ $\dfrac{a^3-2a^2x+2ax^2+x^3}{3a^3+2a^2x-2ax^2+x^3}$

(25.) $\dfrac{16x^4-53x^2+45x+6}{8x^4-30x^3+31x^2-12}$. . . $4x^2-9x+6$ $\dfrac{4x^2+9x+1}{2x^2-3x-2}$

(26.) $\dfrac{24x^5-22x^4-14x^3+24x^2-8x}{18x^5-18x^4-14x^3+30x^2-12x}$. $6x^2-4x$ $\dfrac{4x^3-x^2-3x+2}{3x^3-x^2-3x+3}$

(27.) $\dfrac{x^4+a^2x^2+a^4}{x^4+ax^3-a^3x-a^4}$. . . . x^2+ax+a^2 $\dfrac{x^2-ax+a^2}{x^2-a^2}$

(28.) $\dfrac{9x^5+2x^3+4x^2-x+1}{15x^4-2x^3+10x^2-x+2}$. . $3x^2-x+1$ $\dfrac{3x^3+x^2+1}{5x^2+x+2}$

(29.) $\dfrac{x^4+2x^2+9}{7x^3-11x^2+15x+9}$. . . x^2-2x+3 $\dfrac{x^2+2x+3}{7x+3}$

(30.) $\dfrac{6x^6-x^5-13x^4+11x^3+5x^2-3x-1}{10x^5-19x^4+17x^3-6x^2+x+3}$.

$2x^3-3x^2+x+1$ $\dfrac{3x^3+4x^2-2x-1}{5x^2-2x+3}$

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31. An integer may be expressed in the form of a fraction having any required denominator, by first multiplying by the required denominator, and under the product as numerator writing that denominator. This is merely performing an operation and then indicating the reverse.

EXERCISES.

(1.) Express x with the denominator y. Ans. $\frac{xy}{y}$.

(2.) ... 4 . . . . 7. $\frac{28}{7}$.

(3.) ... $3a$ . . . . $5x$. $\frac{15ax}{5x}$.

(4.) ... $5x$ . . . . $a+x$. $\frac{5ax+5x^2}{a+x}$.

(5.) .. $3ax$ . . . . $a-x$. $\frac{3a^2x-3ax^2}{a-x}$.

(6.) ... $2a^2$ . . . . a^2-x^2. $\frac{2a^4-2a^2x^2}{a^2-x^2}$.

(7.) ... $a+x$ . . . . . . a^2-x^2. $\frac{a^3+a^2x-ax^2-x^3}{a^2-x^2}$.

(8.) ... $a-x$ . . . . . . $a+x$. $\frac{a^2-x^2}{a+x}$.

(9.) ... $3a-1$ . . . . . . $a-2$. $\frac{3a^2-7a+2}{a-2}$.

(10.) ... a^2+ax . . . . . $ax-x^2$. $\frac{a^3x-ax^3}{ax-x^2}$.

(11.) ... a^2+2ax . . . . $(a-x)^2$. $\frac{a^4-3a^2x^2+2ax^3}{(a-x)^2}$.

(12.) ... $(a+x)^2$ . . . . $(a+x)^3$. $\frac{(a+x)^5}{(a+x)^3}$.

---

32. To reduce a mixed quantity to the form of a fraction. Multiply the integral part by the denominator, annexing the numerator with its proper sign; then under the result write the denominator.

EXERCISES.

Reduce the following mixed quantities to the fractional form:

(1.) $1+\frac{x}{a}$. . . . . Ans. $\frac{a+x}{a}$.

(2.) $1-\frac{x}{a}$. . . . . $\frac{a-x}{a}$.

(3.) $a+\frac{2ax+x^2}{a}$. . . . . $\frac{a^2+2ax+x^2}{a}$.

(4.) $a-2x+\frac{x^2}{a}$. . . . . Ans. $\frac{a^2-2ax+x^2}{a}$.

(5.) $x+\frac{a^2-x^2}{x}$. . . . . $\frac{a^2}{x}$.

(6.) $a-\frac{ax+x^2}{a}$. . . . . $\frac{a^2-ax-x^2}{a}$.

(7.) $a+x-\frac{a^2-x^2}{a}$. . . . . $\frac{ax+x^2}{a}$.

(8.) $1+2x-\frac{2+3x}{5}$. . . . . $\frac{3+7x}{5}$.

(9.) $a^2+ax+x^2+\frac{x^3}{a-x}$. . . . . $\frac{a^3}{a-x}$.

(10.) $a^2-ax+x^2-\frac{a^3}{a+x}$. . . . . $\frac{x^3}{a+x}$.

(11.) $2a+3x-\frac{13ax+6x^2}{3a+2x}$. . . . . $\frac{6a^2}{3a+2x}$.

(12.) $2a+3x-\frac{5ax-6x^2}{3a-2x}$. . . . . $\frac{6a^2}{3a-2x}$.

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33. To reduce a fraction to an integral or mixed quantity. Divide the numerator by the denominator if possible, annexing the remainder, if there be any, as in Division.

EXERCISES.

Reduce the following fractions to integrals or mixed quantities:—

(1.) $\frac{ab}{b}$. . . . . Ans. a.

(2.) $\frac{a^2+ax}{a}$. . . . . $a+x$.

(3.) $\frac{a^2-x^2}{a-x}$. . . . . $a+x$.

(4.) $\frac{a^3-2ax^2}{a+x}$. . . . . $a^2-ax-\frac{ax^2}{a+x}$.

(5.) $\frac{a^3+x^3}{a-x}$. . . . . $a^2+ax+x^2+\frac{2x^3}{a-x}$.

(6.) $\frac{a^3+3a^2x+3ax^2}{a+x}$. . . . . . $a^2+2ax+x^2-\frac{x^3}{a+x}$.

34. In order to add together quantities of different denominations, such as pounds and pence; or to subtract the one from the other, so as to express the sum or difference in one number of one denomination; it is evidently necessary first to reduce each of the given quantities to that denomination, after which, the work is accomplished by Simple Addition or Subtraction. In like manner, fractions of different denominators must first be reduced to the same denominator before they can be added, or one subtracted from another. This reduction can always be effected by taking the continued product of all the given denominators as the common denominator, to which each can easily be reduced by multiplying it by all the other denominators. But to preserve the value of each fraction unchanged, the numerator must be multiplied by the same quantity as the denominator. Hence, to reduce fractions to a common denominator, multiply the numerator and denominator of each fraction by all the denominators except its own.*

EXERCISES.

Reduce the following fractions to equivalent ones having a common denominator:—

(1.) $\frac{a}{2}, \frac{a}{3}$. . . . . Ans. $\frac{3a}{6}, \frac{2a}{6}$.

(2.) $\frac{2a}{3}, \frac{4a}{5}$. . . . . $\frac{10a}{15}, \frac{12a}{15}$.

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* 35. The above process may often be much simplified by using the Least Common Multiple of the given denominators, instead of their continued product, for the common denominator. The Least Common Multiple of the denominators being the smallest quantity which contains each of them as a factor, it may be found by the following rule:—

To find the Least Common Multiple of any number of quantities. (1.) Arrange them all in the same line, and if any of them be a factor of any of the others, reject it. (2.) Divide by any factor common to two or more of them, writing the quotients and quantities not divisible by such factor in the second line. (3.) Proceed in the same manner with this second line, and so on, repeating the process as often as possible. Lastly, Find the continued product of all the divisors and quantities found in the last line.

To reduce fractions to a common denominator by this method. Find the Least Common Multiple of the given denominators, for the common denominator, then multiply the numerators respectively by the quotients obtained by dividing this Least Common Multiple by the denominators.

(3.) $\frac{a}{b}, \frac{b}{a}$. . . . . Ans. $\frac{a^2}{ab}, \frac{b^2}{ab}$.

(4.) $\frac{a+x}{4}, \frac{a-x}{6}$. . . . . $\frac{3(a+x)}{12}, \frac{2(a-x)}{12}$.

(5.) $\frac{1}{x^2}, \frac{1}{y^2}, \frac{2}{xy}$. . . . . $\frac{y^2}{x^2y^2}, \frac{x^2}{x^2y^2}, \frac{2xy}{x^2y^2}$.

(6.) $\frac{1}{a}, \frac{1}{ab}, \frac{1}{abc}$. . . . . $\frac{bc}{abc}, \frac{c}{abc}, \frac{1}{abc}$.

(7.) $\frac{2x-3}{4a}, \frac{3x-2}{6ax}$. . . . . . . $\frac{6x^2-9x}{12ax}, \frac{6x-4}{12ax}$.

(8.) $\frac{x}{x+y}, \frac{y}{x-y}$. . . . . $\frac{x(x-y)}{x^2-y^2}, \frac{y(x+y)}{x^2-y^2}$.

(9.) $\frac{a}{a-b}, \frac{a-b}{b}, \frac{a-b}{a+b}$. . $\frac{ab\,(a+b)}{b(a^2-b^2)}, \frac{(a-b)\,(a^2-b^2)}{b(a^2-b^2)}, \frac{b(a-b)^2}{b(a^2-b^2)}$.

(10.) $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$. . . . . $\frac{bc}{abc}, \frac{ac}{abc}, \frac{ab}{abc}$.

(11.) $\frac{x+y}{x-y}, \frac{x-y}{x+y}$. . . . $\frac{(x+y)^2}{x^2-y^2}, \frac{(x-y)^2}{x^2-y^2}$.

(12.) $\frac{a}{a+x}, \frac{x}{a-x}, \frac{x}{a+x}$. $\frac{a(a-x)}{a^2-x^2}, \frac{x(a+x)}{a^2-x^2}, \frac{x(a-x)}{a^2-x^2}$.

(13.) $\frac{x^2+y^2}{x^2-y^2}, \frac{x}{x+y}, \frac{y}{y-x}$. $\frac{x^2+y^2}{x^2-y^2}, \frac{x(x-y)}{x^2-y^2}, \frac{-y(x+y)}{x^2-y^2}$.

(14.) $\frac{1}{x+1}, \frac{2}{x+2}, \frac{3}{x+3}, \frac{4}{x+4}$.

(15.) $\frac{1}{x-1}, \frac{2}{2x-1}, \frac{3}{3x-1}, \frac{4}{4x-1}$.

ADDITION OF FRACTIONS.

36. To add fractions. Reduce them if necessary to equivalent fractions having the same denominator; then add the numerators, and under the sum write the common denominator.

EXERCISES.

(1.) $\frac{a}{2}+\frac{b}{2}$. . . . . Ans. $\frac{a+b}{2}$.

(2.) $\frac{a}{3}+\frac{b}{3}+\frac{c}{3}$. . . . . $\frac{a+b+c}{3}$.

(3.) $\frac{a+b}{5}+\frac{a-b}{5}$. . . . . Ans. $\frac{2a}{5}$.

(4.) $\frac{x}{7}+\frac{2x-1}{7}+\frac{3x-2}{7}$. . . . . . . $\frac{6x-3}{7}$.

For additional exercises, add the fractions in the last set.

Answers. (1.) $\frac{5a}{6}$. (2.) $\frac{22a}{15}$. (3.) $\frac{a^2+b^2}{ab}$. (4.) $\frac{5a+x}{12}$.

(5.) $\frac{x^2+2xy+y^2}{x^2y^2}$. (6.) $\frac{bc+c+1}{abc}$. (7.) $\frac{6x^2-3x-4}{12ax}$. (8.) $\frac{x^2+y^2}{x^2-y^2}$.

(9.) $\frac{a^3+a^2b-2ab^2+2b^3}{b(a^2-b^2)}$. (10.) $\frac{ab+ac+bc}{abc}$. (11.) $\frac{2(x^2+y^2)}{x^2-y^2}$.

(12.) $\frac{a}{a-x}$. (13.) $\frac{2x}{x+y}$. (14.) $\frac{10x^3+70x^2+150x+96}{x^4+10x^3+35x^2+50x+24}$.

(15.) $\frac{96x^3-150x^2+70x-10}{24x^4-50x^3+35x^2-10x+1}$.

SUBTRACTION OF FRACTIONS.

37. To subtract one fraction from another. Reduce them if necessary to the same denominator; then subtract the one numerator from the other, and under the difference write the common denominator.

EXERCISES.

(1.) $\frac{2a}{3}-\frac{a}{3}$. . . . . Ans. $\frac{a}{3}$.

(2.) $\frac{a}{2}-\frac{b}{2}$. . . . . $\frac{a-b}{2}$.

(3.) $\frac{a}{b}-\frac{b}{a}$. . . . . $\frac{a^2-b^2}{ab}$.

(4.) $\frac{x}{x-y}-\frac{x}{x+y}$. . . . . $\frac{2xy}{x^2-y^2}$.

(5.) $\frac{x+y}{x-y}-\frac{x-y}{x+y}$. . . . . $\frac{4xy}{x^2-y^2}$.

(6.) $\frac{2}{1-x^2}-\frac{1}{1-x}$. . . . . $\frac{1}{1+x}$.

(7.) $\frac{x^2+y^2}{x^2-y^2}-\frac{y}{x-y}$. . . . . . . $\frac{x}{x+y}$.

(8.) $\frac{x}{x+1} - \frac{x}{x+2}$. . . . . . . Ans. $\frac{x}{x^2+3x+2}$.

(9.) $x + \frac{x}{x-2} - \left(x + \frac{x}{x-1}\right)$. . . $\frac{x}{x^2-3x+2}$.

(10.) $\frac{3+2x}{1+2x+x^2} - \frac{2}{1+x}$. . . . . . . $\frac{1}{1+2x+x^2}$.

MULTIPLICATION OF FRACTIONS.

38. To multiply fractions. Multiply all the numerators together for the numerator, and all the denominators for the denominator, suppressing such factors as are common to both numerator and denominator.

EXERCISES.

(1.) $\frac{a}{2} \times 2$. . . . . Ans. a.

(2.) $\frac{a}{b} \times b$. . . . . b.

(3.) $\frac{3x}{2} \times \frac{2x}{3}$. . . . . x^2.

(4.) $\frac{2-x}{3} \times \frac{1-2x}{2}$. . . . . $\frac{2-5x+2x^2}{6}$.

(5.) $\frac{a^2-x^2}{a} \times \frac{a}{a-x} \times \frac{a}{a+x}$. . . a.

(6.) $\frac{a^2+x^2}{a^2-x^2} \times \frac{a-x}{a+x}$. . . . . . . $\frac{a^2+x^2}{(a+x)^2}$.

(7.) $\frac{a^2-2ab+b^2}{a^2+2ab+b^2} \times \frac{a+b}{a-b}$. . . . . $\frac{a-b}{a+b}$.

(8.) $\frac{a^2+ab+b^2}{a^2-ab+b^2} \times \frac{a-b}{a+b}$. . . . . $\frac{a^3-b^3}{a^3+b^3}$.

(9.) $\frac{x^2+3x+2}{x^2+2x+1} \times \frac{x^2+5x+4}{x^2+7x+12}$. . . $\frac{x+2}{x+3}$.

(10.) $\frac{1+x}{1-x} \times 1-x$. . . . . $1+x$.

(11.) $\left(\frac{x}{2} + \frac{2x}{3}\right)6$. . . . . $7x$.

(12.) $\left(\frac{a}{3} + \frac{b}{4} + \frac{c}{5}\right) \times 60$. . . . . $20a+15b+12c$.

(13.) $\left(\frac{a}{2}+\frac{a}{4}+\frac{a}{6}\right)\times$ L. C. M. of denominators. Ans. $11a$.

(14.) $\left(\frac{x}{a+x}+\frac{a}{a-x}-\frac{a^2+x^2}{a^2-x^2}\right)\times$ L. C. M. „ . . . $2ax-2x^2$.

DIVISION OF FRACTIONS.

39. To divide one fraction by another. Invert the divisor, and proceed as in Multiplication.

EXERCISES.

(1.) $\frac{3a}{5}\div\frac{9a}{10}$. . . . . Ans. $\frac{2}{3a}$.

(2.) $\frac{5ax}{y}\div\frac{2x}{ay}$. . . . . $\frac{5a^2}{2}$.

(3.) $\frac{2-x}{y}\div\frac{2-x}{x}$. . . . . $\frac{x}{y}$.

(4.) $1\div\frac{1}{x}$. . . . . x.

(5.) $\frac{1}{x}\div\frac{1}{x^2}$. . . . . x.

(6.) $\frac{1}{1+x}\div\left(1-\frac{1}{1+x}\right)$. . . . . $\frac{1}{x}$.

(7.) $\frac{a^2-ab+b^2}{a^2+ab+b^2}\div\frac{a-b}{a+b}$. . . . . $\frac{a^3+b^3}{a^3-b^3}$.

(8.) $\frac{a^2+x^2}{a^2-x^2}\div\frac{a+x}{a-x}$. . . . . $\frac{a^2+x^2}{(a+x)^2}$.

(9.) $\frac{a^3-ax^2}{x^2}\div\frac{ax-x^2}{a}$. . . . . $\frac{a^2(a+x)}{x^3}$.

(10.) $\left(1+\frac{x^2}{a^2-x^2}\right)\div\left(1-\frac{a^2}{a^2-x^2}\right)$. . . $-\frac{a^2}{x^2}$.

(11.) $\frac{x^2+4x+3}{x^2+5x+6}\div\frac{x^2+2x+1}{x^2+4x+4}$. . . . . . $\frac{x+2}{x+1}$.

(12.) $\frac{x^2-2x-3}{x^2+x-12}\div\frac{x^2-2x-8}{x^2+x-2}$, ... $\frac{x^2-1}{x^2-16}$.

(13.) $\left(\frac{a+1}{a-1}+\frac{a-1}{a+1}\right)\div\left(\frac{a+1}{a-1}-\frac{a-1}{a+1}\right)$. . . $\frac{a^2+1}{2a}$.

(14.) $\left(\frac{a+b}{a-b}+\frac{a-b}{a+b}\right)\div\left(\frac{a+b}{a-b}-\frac{a-b}{a+b}\right)$. . . $\frac{a^2+b^2}{2ab}$.

CHAPTER IV.

EQUATIONS OF THE FIRST DEGREE.

40. An equation is a statement of equality between two quantities. Thus $a + x = b - x$ is an equation of which $a + x$ is the one side or member, and $b - x$ the other.

In this equation it is asserted that the unknown quantity x is so connected with the two known quantities a and b, that the sum of a and x is the same as the excess of b above x.

A *simple equation* is one which contains only one power of the unknown quantity; if it contain more than one, it is compound.

An equation is said to be of the first, second, third, or nth degree, according as the highest power of the unknown quantity in it is the first, second, third, or nth power.

To solve an equation, or to find the value of the unknown quantity, it is necessary to separate it from all the other quantities with which it may be connected, so that it may stand alone on one side of the equation and its value on the other.

Axiom 1. If equals, or the same be added to equals, the wholes are equal.

2. If equals, or the same be taken from equals, the remainders are equal.

3. If equals be multiplied by the same or by equals, the products are equal.

4. If equals be divided by the same or by equals, the quotients are equal.

General Principles.—1. Any quantity may be transposed from the one side of an equation to the other by changing its sign.

2. A coefficient or multiplier may be removed by dividing both sides of the equation by it.

3. An equation may be cleared of fractions by multiplying both sides by all the denominators, or by their least common multiple.

To solve the following equations:—1. Clear the equation of fractions if necessary. 2. Transpose all the terms which contain the unknown quantity to the one side, and all the known terms to the other, incorporating like quantities. 3. Divide both sides by the coefficient of the unknown quantity.*

(1.) $x - 2 = 3$. . . . . Ans. $x = 5$.
(2.) $x + 2 = 3$. . . . . $x = 1$.
(3.) $x - 1 + 2 = 4$. . . . . $x = 3$.
(4.) $x + 1 - 2 = 3 - 1$. . . . . $x = 3$.
(5.) $2x - 4 = 6$. . . . . $x = 5$.
(6.) $3x + 1 = 4 + 3$. . . . . $x = 2$.
(7.) $4x - 2 = 3x + 1$. . . . . $x = 3$.
(8.) $6x + 3 = 5x + 4$. . . . . $x = 1$.
(9.) $6x - 5 = 4x + 17$. . . . . $x = 11$.
(10.) $7x - 9 = 3x + 7$. . . . . $x = 4$.
(11.) $9x - 7 = 9 + 7x$. . . . . $x = 8$.
(12.) $11x + 1 = 7x + 21$. . . . . $x = 5$.
(13.) $5x - 3x = 12 - x$. . . . . $x = 4$.
(14.) $4x - 11 + 5 = 29 - x$. . . . . $x = 7$.
(15.) $7x - 9 + 1 = 8x - 11$. . . . . $x = 3$.
(16.) $17 - 3x + 2 = 4x - 9$. . . . . $x = 4$.
(17.) $3x - 4 = 7 + 4x - 12$. . . . . $x = 1$.
(18.) $114 = 7x - 11 + 4x + 4$. . . . . . . $x = 11$.
(19.) $8x - 3x + 2 = 11x - 4x - 8$. . . . $x = 5$.
(20.) $11x - 7x + 4x = 3x + 40$. . . . . . $x = 8$.

(21.) $3x + \frac{1}{2} = 5x - 3\frac{1}{2}$. . . . . $x = 2$.

(22.) $7x - 2\frac{1}{2} = 2\frac{1}{2} + 6x$. . . . . $x = 5$.

(23.) $x + \frac{x}{2} = 9$. . . . . $x = 6$.

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* Example. Given $\frac{4x}{5} - \frac{2x}{3} = \frac{2}{5}$. Multiplying by 5 and by 3,

we get . . . $12x - 10x = 6$,

or . . . . $2x = 6$; and dividing by 2,

we get . . . . . $x = 3$.

(24.) $x - \frac{x}{3} = 10 - x$. . . . . Ans. $x = 6$.

(25.) $4x + \frac{x}{4} = 3x + 5$. . . . . $x = 4$.

(26.) $\frac{5x}{3} + \frac{2}{3} = 2x - 1$. . . . . $x = 5$.

(27.) $\frac{x}{2} + \frac{x}{3} = x - 2$. . . . . $x = 12$.

(28.) $\frac{x}{2} - 6 = 6 - \frac{x}{4}$. . . . . $x = 16$.

(29.) $\frac{2x}{5} - \frac{3x}{10} = \frac{x}{2} - 8$. . . . . $x = 20$.

(30.) $x - \frac{x}{2} - \frac{x}{3} - \frac{1}{6} = 1$. . . . . $x = 7$.

(31.) $\frac{4x}{7} - \frac{2x}{3} + \frac{7x}{3} = 2x + 5$. . . . . $x = 21$.

(32.) $\frac{x}{4} + \frac{3}{4} = \frac{3x}{7} - \frac{7}{4}$. . . . . $x = 14$.

(33.) $\frac{x}{2} - \frac{x}{3} - \frac{x}{4} + \frac{5}{4} = \frac{3}{4}$. . . . . $x = 6$.

(34.) $\frac{3x}{2} - \frac{2x}{3} + \frac{x}{6} = 4\frac{1}{2} - \frac{x}{2}$. . . . . $x = 3$.

(35.) $\frac{x}{2} - \frac{x}{3} + \frac{x}{4} - \frac{x}{5} = 13$. . . . . $x = 60$.

(36.) $\frac{x}{2} - \frac{x}{4} - \frac{x}{8} = 4\frac{1}{8} - \frac{x}{4}$. . . . . $x = 11$.

(37.) $\frac{3x}{8} + \frac{7x}{18} - \frac{2x}{3} = \frac{x}{6} - \frac{5}{4}$. . . . . $x = 18$.

(38.) $\frac{3x}{4} + \frac{4x}{5} - \frac{8x}{15} = \frac{x}{60} + 40$. . . . . . . $x = 40$.

(39.) $\frac{10x}{3} + \frac{7x}{5} + \frac{x}{2\frac{1}{2}} = \frac{4x}{1\frac{1}{2}} + 7\frac{2}{5}$. . . . . . . $x = 3$.

(40.) $\frac{x}{4} - 4\frac{1}{2} + \frac{x}{5\frac{1}{2}} + \frac{x}{2} = \frac{16\frac{1}{4}}{5\frac{1}{2}}$. . . . . . $x = 8$.

(41.) $3(4 + x) = 2(15 - x) - x$. . . . . . . $x = 3$.

(42.) $4x - 3(4 + x) = 6$. . . . . $x = 18$.

(43.) $2x - 3(2 - x) = 4$. . . . . . . Ans. $x = 2$.

(44.) $4(2 - x) - 3(3 - 2x) = 9$. . . . . . $x = 5$.

(45.) $6(x - 3) = 9(x - 4) - (x - 6)$. . . . $x = 6$.

(46.) $\frac{2x - 1}{2} = 11\frac{1}{2} - x$. . . . . $x = 6$.

(47.) $\frac{3x}{7} + x = 11 - \frac{x - 2}{5}$. . . . . $x = 7$.

(48.) $\frac{2x + 1}{3} - \frac{3x + 1}{4} = \frac{x}{2} - \frac{3x + 2}{6}$. . . . . . $x = 5$.

(49.) $\frac{2x - 1}{4} - \frac{3x - 1}{5} = \frac{x - 1}{4} - \frac{x + 2}{5}$. . . $x = 4$.

(50.) $\frac{3}{x} - \frac{2}{x} + 1 = \frac{4}{x} + \frac{1}{4}$. . . . . $x = 4$.

(51.) $\frac{2}{3x} + \frac{3}{2x} = \frac{1}{6}$. . . . . $x = 13$.

(52.) $\frac{3x - 2}{12} + \frac{2x - 3}{x + 1} = \frac{x + 4}{4}$. . . . . . . $x = 5$.

(53.) $\frac{3x + 7}{12} - \frac{x + 7}{5x - 11} = \frac{x}{4}$. . . . . $x = 7$.

(54.) $\frac{x - 5}{x + 5} - \frac{x + 5}{x - 5} = \frac{4x - 25}{x + 5} - 4$. . . . . . $x = 9$.

(55.) $\frac{7x + 16}{21} - \frac{x + 8}{4x - 11} = \frac{x}{3}$. . . . . $x = 8$.

(56.) $\frac{x - 7}{x + 7} + \frac{1}{2(x + 7)} = \frac{2x - 15}{2x - 6}$. . . . . . . $x = 8$.

(57.) $\frac{1}{7}(x - \frac{1}{2}) - \frac{1}{5}\left(\frac{2}{3} - x\right) = \frac{43}{30}$. . . . . $x = 4\frac{7}{9}$.

(58.) $\frac{1}{4}\left(4 + \frac{3x}{2}\right) - \frac{1}{7}\left(2x - \frac{1}{3}\right) = \frac{31}{28}$. . . . $x = \frac{2}{3}$.

(59.) $\frac{1}{5}\left(\frac{1}{2} + \frac{2}{x}\right) - \frac{1}{7}\left(\frac{3}{4} - \frac{4}{x}\right) = \frac{1}{5}\left(\frac{7}{x} - \frac{9}{4x}\right)$. . $x = 3$.

(60.) $\frac{1}{2}\left(\frac{2}{3}x - 3\right) - \frac{1}{3}\left(\frac{3}{4}x - 4\right) = \frac{1}{4}\left(\frac{4}{5}x - 1\frac{3}{5}\right)$. $x = 2$.

(61.) $ax = b.$ . . . . Ans. $x = \frac{b}{a}.$

(62.) $\frac{a}{x} = b.$ . . . . $x = \frac{a}{b}.$

(63.) $\frac{x}{a} = b.$ . . . . $x = ab.$

(64.) $\frac{x}{a} + \frac{x}{b} - c.$ . . . . $x = \frac{abc}{a+b}.$

(65.) $\frac{x}{a} - \frac{x}{b} = c.$ . . . . $x = \frac{abc}{b-a}.$

(66.) $\frac{a}{x} + \frac{b}{x} = c.$ . . . . $x = \frac{a+b}{c}.$

(67.) $\frac{a}{x} - \frac{b}{x} = c.$ . . . . $x = \frac{a-b}{c}.$

(68.) $\frac{ax}{b} + \frac{cx}{d} = e.$ . . . . $x = \frac{bde}{ad+bc}.$

(69.) $\frac{a}{bx} - \frac{c}{dx} = e.$ . . . . $x = \frac{ad-bc}{bde}.$

(70.) $\frac{a}{bx} + \frac{b}{ax} = a^2 + b^2.$ . . . . . $x = \frac{1}{ab}.$

(71.) $\frac{2a+3x}{x} + \frac{3a-4x}{x} = \frac{4a+5x}{x}.$ . . . $x = \frac{a}{6}.$

(72.) $2x - 3a - bx = ab - 6x.$ . . . . $x = \frac{a(3+b)}{8-b}.$

(73.) $x + \frac{ax}{b} + \frac{cx}{b} = d.$ . . . . . . $x = \frac{bd}{a+b+c}.$

(74.) $\frac{ax-2b}{2} + \frac{bx-2a}{2} = a - b.$ . . . $x = \frac{4a}{a+b}.$

(75.) $\frac{a}{1-x} - \frac{a}{1+x} = \frac{b}{1-x}.$ . . . . . $x = \frac{b}{2a-b}.$

(76.) $\frac{a-x}{b} + \frac{b-x}{a} = 2.$ . . . . $x = \frac{(a-b)^2}{a+b}.$

(77.) $\frac{ab+bx}{a} - \frac{ab-ax}{b} = 2x.$ . . . . $x = \frac{ab}{a-b}.$

(78.) $x + a : x - a :: a + 2b : a - 3b.$. $x = a\frac{2a-b}{5b}.$

Ans.

(79.) $4x - a : 2x - b :: 2x + b : x + a.$ $x = \frac{a^2 - b^2}{3a}.$

(80.) $x+a:x+b::(2x+a+c)^2:(2x+b+c)^2.$ $x = \frac{c^2 - ab}{a + b - 2c}.$

SIMPLE EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

41. When one unknown quantity occurs in the expression for another unknown quantity, it is plain that the value of neither of them can be determined, unless some other condition be given.

In order to limit such questions, there must always be as many conditions or independent equations given as there are unknown quantities to be determined; and as these equations coexist, or are all true at the same time, they are called *simultaneous equations.*

To solve such equations it is necessary to *eliminate* one of the unknown quantities, that is, to derive from the given equations another equation in which one of the unknown quantities does not occur, and which shall therefore contain only one unknown quantity.

The best methods of elimination are the following:—

First, If the coefficients of the quantity to be eliminated be the same in both equations, take the difference or sum of the equations, according as the coefficients have the same or contrary signs, and the quantity will be eliminated. Thus,

Given $2x + 3y = 22$
and $5x - 3y = 13.$

By Addition we get $7x = 35$; an equation from which y is eliminated.

If the coefficients of one of the quantities be not the same in both equations, multiply one or both equations by such numbers as will make them the same, and proceed as above.

Second Method.—Find an expression for one of the unknown quantities in both equations, and put the one expression equal to the other. Thus,

Given $2x + 3y = 22$
and $5x - 3y = 13$. From the first equation we get $y = \frac{22 - 2x}{3}$, and from the second we get $y = \frac{5x - 13}{3}$. Lastly, by putting the one of these values of y equal to the other,

$$\text{we get } \frac{5x - 13}{3} = \frac{22 - 2x}{3},$$

$$\text{or} \quad 7x = 35.$$

Third Method.—Find a value for one of the quantities in one of the equations, and substitute that value instead of it in the other equation. Thus,

Given $2x + 2y = 22$
and $5x - 3y = 13$. From the first we get $y = \frac{22 - 2x}{3}$, and by substitution in the second, we get $5x - 3.\frac{22 - 2x}{3} = 13$,

$$\text{or} \quad 7x = 35.$$

Having eliminated one of the unknown quantities by any of the above methods, the value of the other is found from the resulting equation; and by substituting this value in either of the given equations, the value of the remaining quantity may also be found; or, the value of this latter may be found by resuming the original equations, and from them eliminating the other quantity.

EXERCISES.

(1.) $x + y = 7$. . . . . Ans. $x = 5$.
$x - y = 3$. . . . . $y = 2$.
(2.) $x + y = 11$. . . . . $x = 2$.
$y - x = 7$. . . . . $y = 9$.
(3.) $2x + 3y = 16$. . . . . $x = 5$.
$3x - 3y = 9$. . . . . $y = 2$.
(4.) $x + 2y = 8$. . . . . $x = 2$.
$y + 2x = 7$. . . . . $y = 3$.

(5.) $x + y = 6.$ . . . . Ans. $x = 4.$
$3x - 4y = 4.$ . . . . $y = 2.$

(6.) $x - y = 2.$ . . . . $x = 3.$
$4x + 3y = 15.$ . . . . $y = 1.$

(7.) $4x + 3y = 41.$ . . . . $x = 5.$
$2x - y = 3.$ . . . . $y = 7.$

(8.) $5x - 3y = 11.$ . . . . $x = 4.$
$3x - 3y = 3.$ . . . . $y = 3.$

(9.) $3x + 4y = 11.$ . . . . $x = 1.$
$15x - 2y = 11.$ . . . . $y = 2.$

(10.) $4x - 7y = 5.$ . . . . $x = 3.$
$7x - 4y = 17.$ . . . . $y = 1.$

(11.) $17x - 5y = 99.$ . . . . $x = 7.$
$7y - 4x = 0.$ . . . . $y = 4.$

(12.) $11x - 8y = 7.$ . . . . $x = 5.$
$9y - 7x = 19.$ . . . . $y = 6.$

(13.) $\frac{1}{2}x + \frac{1}{2}y = 6.$ . . . . $x = 9.$
$\frac{1}{2}x - \frac{1}{2}y = 3.$ . . . . $y = 3.$

(14.) $\frac{1}{2}x + \frac{1}{3}y = 9.$ . . . . $x = 10.$
$\frac{1}{5}x - \frac{1}{4}y = -1.$ . . . . $y = 12.$

(15.) $\frac{2}{3}x + \frac{3}{4}y = 10.$ . . . . $x = 6.$
$\frac{1}{2}y - \frac{1}{2}x = 1.$ . . . . $y = 8.$

(16.) $\frac{1}{4}x + \frac{1}{3}y = 3\frac{7}{12}.$ . . . . $x = 5.$
$\frac{2}{3}x - \frac{1}{4}y = 1\frac{7}{12}.$ . . . . $y = 7.$

(17.) $\frac{3x - y}{4} + 2y = 24.$ . . . . $x = 11.$
$\frac{3y - x}{4} + 2x = 26.$ . . . . $y = 9.$

(18.) $\frac{4x + 2y}{5} = 10.$ . . . . $x = 6.$
$\frac{4y - 2x}{5} = 8.$ . . . . $y = 13.$

(19.) $\frac{x + 1}{2} - \frac{y + 1}{3} = 1.$ . . . . $x = 7.$
$\frac{x + 2}{3} + \frac{y + 2}{2} = 8.$ . . . . $y = 8.$

(20.) $\dfrac{x+2}{y} = \frac{2}{3}$. . . . . Ans. $x = 4$.

$\dfrac{y-1}{3x} = \frac{2}{3}$. . . . . $y = 9$.

(21.) $x + y = s$. . . . . . . Ans. $x = \frac{1}{2}(s + d)$.

$x - y = d$. . . . . $y = \frac{1}{2}(s - d)$.

(22.) $ax + by = c$. . . . . $x = \dfrac{b^2 + c}{a + b}$.

$x - y = b$. . . . . $y = \dfrac{c - ab}{a + b}$.

(23.) $x + ay = b$. . . . . $x = \dfrac{ac - b}{a^2 - 1}$.

$y + ax = c$. . . . . $y = \dfrac{ab - c}{a^2 - 1}$.

(24.) $x + y = a$. . . . . $x = \dfrac{ab + b}{a + b}$.

$ax - by = b$. . . . . $y = \dfrac{a^2 - b}{a + b}$.

(25.) $x - y = a$. . . . . $x = \dfrac{3a + b}{7}$.

$4x + 3y = b$. . . . . $y = \dfrac{b - 4a}{7}$.

(26.) $\dfrac{x}{a} + \dfrac{y}{a} = b$. . . . . $x = \frac{1}{2}a(b + c)$.

$\dfrac{x}{a} - \dfrac{y}{a} = c$. . . . . $y = \frac{1}{2}a(b - c)$.

(27.) $\dfrac{x}{a} + \dfrac{y}{b} = c$. . . . . $x = ad\dfrac{bc + ef}{ae + bd}$.

$\dfrac{x}{d} - \dfrac{y}{e} = f$. . . . . $y = be\dfrac{ac - df}{ae + bd}$.

(28.) $\dfrac{ax + by}{c} = 2c$. . . . . $x = 2ac\dfrac{c - b}{a^2 + b^2}$.

$\dfrac{ay - bx}{c} = 2a$. . . . . $y = 2c\dfrac{a^2 + bc}{a^2 + b^2}$.

(29.) $\dfrac{x+1}{a} - \dfrac{y+1}{b} = c$. $x = \dfrac{ab^2c + a^2bd - 2a^2 - b^2 - ab}{a^2 + b^2}$.

$\dfrac{x+2}{b} + \dfrac{y+2}{a} = d$. $y = \dfrac{ab^2d - a^2bc - a^2 - 2b^2 - ab}{a^2 + b^2}$.

(30.) $\dfrac{x+2}{y} = \dfrac{a}{b}$. . . Ans. $x = b\dfrac{a-2b}{b^2-a^2} = b\dfrac{2b-a}{a^2-b^2}$.

$\dfrac{y-1}{x} = \dfrac{a}{b}$. . . . $y = b\dfrac{b-2a}{b^2-a^2} = b\dfrac{2a-b}{a^2-b^2}$.

42. If the given equations contain three or more unknown quantities, the elimination may be effected on the same general principles. Thus, let $x + 2y + 3z = 20$; $2x - y + 3z = 13$; and $3x + 4y - 2z = 10$, be the given equations, by subtracting the second of these from the first, z is eliminated, and the result is $-x + 3y = 7$. Again, z may be eliminated between the second and third, by multiplying the second by 2, and the third by 3, and adding the products; the result is $13x + 10y = 56$; and having thus obtained two equations containing only two unknown quantities, their values may readily be found as in the examples already given.

In this case we have eliminated z between the first and second equations, and then between the second and third; but we might have eliminated it with almost equal facility between the first and second, and the first and third; or between the first and third, and the second and third. It is obvious also that we might commence by eliminating either x or y, instead of z; and each of these cases would admit of the same three variations.

EXERCISES.

(1.) $x + y + z = 9$. . . . . Ans. $x = 2$.
$x - y + z = 3$. . . . . $y = 3$.
$x + y - z = 1$. . . . . $z = 4$.

(2.) $2x - 3y + 4z = 7$. . . . . $x = 3$.
$3x - 2y + 2z = 7$. . . . . $y = 5$.
$4x - 2y + 3z = 14$. . . . . $z = 4$.

(3.) $4x + 3y - 2z = 3$. . . . . $x = 1$.
$4x - 3y + z = 0$. . . . . $y = 3$.
$2x - 2y + 3z = 11$. . . . . $z = 5$.

(4.) $2x - 5y + 3z = -1$. . . . . $x = 6$.
$3x + 6y - 5z = 28$. . . . . $y = 5$.
$5x - 4y - z = 6$. . . . . $z = 4$.

(5.) $x + y + z = 15.$ . . . . . . Ans. $x = 5.$
$2x - 5y + 3z = 16.$ . . . . $y = 3.$
$4x + 3y - 5z = -6.$ . . . . $z = 7.$

(6.) $7x - 4y - 2z = 5.$ . . . . $x = 5.$
$3x - 5y + 7z = -13.$ . . . . $y = 7.$
$4x + 2y - 3z = 31.$ . . . . $z = 1.$

(7.) $x + y + z + w = 14.$ . . . . $x = 4.$
$2x - y + z + w = 12.$ . . . . $y = 3.$
$3x + 2y - z - w = 11.$ . . . . $z = 2.$
$-x + 3y - z + 2w = 13.$ . . . . . . $w = 5.$

(8.) $\frac{1}{2}x - \frac{1}{2}y + \frac{1}{2}z = 1.$ . . . . $x = 3.$
$\frac{1}{3}x + \frac{3}{4}y - \frac{1}{8}w = 4.$ . . . . $y = 5.$
$\frac{1}{4}x + \frac{1}{2}z + \frac{3}{8}w = 5.$ . . . . $z = 4.$
$\frac{1}{5}y - \frac{1}{4}z + \frac{1}{3}w = 2.$ . . . . $w = 6.$

QUESTIONS PRODUCING SIMPLE EQUATIONS.

(1.) What number is that whose double diminished by 5 is equal to 7? Ans. 6.

(2.) What number is that whose half and third part together are equal to 10? Ans. 12.

(3.) Find a number such that if 7 be taken from its double, the remainder shall be equal to 11. Ans. 9.

(4.) Find a number, from whose double if 3 be taken, the remainder shall be equal to the required number increased by 2. Ans. 5.

(5.) Find a number the third part of which exceeds the fourth part by 3. Ans. 36.

(6.) There is a number, from which if 6 be taken, and the remainder multiplied by 6, the product is the same as that obtained by subtracting 4 from the number, and multiplying the remainder by 4. Find the number. Ans. 10.

(7.) Find a number such that if its sixth part be added to the seventh part of its double, the sum shall be $3\frac{5}{6}$ less than the number itself. Ans. 7.

(8.) Find a number such that if from the sum of its third, fourth, and fifth parts its half be taken, the remainder shall be 17. Ans. 60.

(9.) Divide 40 into two parts such that if one-tenth of the less be taken from one-fifth of the greater, the remainder is 5. Ans. 10 and 30.

(10.) Divide 28 into two parts, of which the one shall be three-fourths of the other. Ans. 12 and 16.

(11.) Divide 36 into two parts having the ratio of 7 to 11. Ans. 14 and 22.

(12.) Divide £5 among three persons, giving the first 5s. more than the second, and the second 10s. more than the third. Ans. The shares are 40s., 35s., and 25s.

(13.) Divide 2 guineas among three persons, giving the first double of the second, and the third as much as them both. Ans. 7s., 14s., and 21s.

(14.) Divide £1 among four persons, giving the first 1s. more than the second, the second 1s. more than the third, and the third 1s. more than the fourth.

Ans. 3s. 6d., 4s. 6d., 5s. 6d., and 6s. 6d.

(15.) The sum of exactly the same number of half-crowns, shillings, and sixpences is £1; how many is there of each?

Ans. 5.

(16.) I have exactly the same number of sovereigns, half-sovereigns, shillings, and sixpences; how many is there of each, the amount being 6 guineas? Ans. 4.

(17.) In paying an account of £6 in half-sovereigns and half-crowns, how many of each must be used, so that the number of the latter shall be double that of the former?

Ans. 8 and 16.

(18.) My purse and money together are worth £1, and the money is four times the value of the purse; how much is there in it? Ans. 16s.

(19.) My coat and hat together are worth £3, 10s., and the hat is one-fourth the value of the coat; find the value of each. Ans. The hat is worth 14s., and the coat, 56s.

(20.) A boy's age is one-fourth of his father's, and he has a brother one-fifth of his own age; the ages of the three amount to 52: find the age of each.

Ans. 40, 10, and 2.

(21.) A's age is double of B's, and B's is triple of C's; the sum of all their ages is 100: what is the age of each?

Ans. A's 60, B's 30, C's 10.

(22.) A son's age is one-fourth of his father's, but three years ago it was one-sixth; what is the age of each?

Ans. Son's, $7\frac{1}{2}$; father's, 30.

(23.) A man at the time of his marriage was twice as old as his wife, but after 20 years her age was two-thirds of his; what were their ages at marriage?

Ans. The woman's 20, and the man's 40.

(24.) The ages of two brothers differ by 2 years, and when added together amount to the age of their father; but if the father's age be increased by half that of the elder brother and one-third that of the younger, it will amount to 71: what is the age of each? Ans. 24, 26, and 50.

(25.) A boy is one-third the age of his father, and has a brother one-sixth of his own age; what is the age of each, the ages of all three amounting to 50?

Ans. 36, 12, and 2.

(26.) The age of one of two brothers is double that of the other, and the one is as much above 12 as the other is below it; what is the age of each? Ans. 8 and 16.

(27.) A's age is double that of B's, and 10 years ago he was three times as old; find A's age. Ans. 40.

(28.) Five years ago A's age was three times that of B, but in five years more he will be only twice as old; what is A's age? Ans. 35.

(29.) The ages of A and B together amount to 100, but if B's age be doubled it will exceed A's by five years; find the age of each. Ans. A's 65, and B's 35.

(30.) A shepherd had in his flock four times as many sheep as lambs; but having sold fourscore sheep, and bought onescore of lambs, the number of the former is now only three times that of the latter; how many of each had he at first? Ans. 28 score of sheep, and 7 score of lambs.

(31.) A post is one-fourth of its length in mud, one-third in water, and 10 feet above water; what is its whole length?

Ans. 24 feet.

(32.) Having paid away one-fourth of my money, and then two-fifths of what remained, I had £54 left; how much had I at first? Ans. £120.

(33.) Having paid away half of my money, two-thirds of the remaining half, and three-fourths of what then remained, I had £10 left; what had I at first? Ans. £240.

(34.) A class consists of three times as many boys as girls; but when three of each have left, there remain four times as many boys as girls: how many of each are in the class? Ans. 27 boys and 9 girls.

(35.) The product of two numbers is 80; but if the greater

be increased by 1, the product is increased by 8: what are the numbers? Ans. 10 and 8.

(36.) Find two consecutive numbers such that three-fourths of the less shall be equal to two-thirds of the greater. Ans. 8 and 9.

(37.) Find two consecutive numbers such that the sum of the half and fifth parts of the first shall be equal to the sum of the third and fourth parts of the second.

Ans. 5 and 6.

(38.) Find a fraction such that if 3 be added to its numerator it becomes $\frac{1}{2}$, but if 1 be taken from its denominator it becomes $\frac{1}{3}$. Ans. $\frac{5}{16}$.

(39.) Find a fraction such that if 1 be added to the denominator, or if 1 be taken from both numerator and denominator, the resulting fractions are each $= \frac{1}{2}$. Ans. $\frac{7}{13}$.

(40.) Required a fraction such that if its numerator be added to both numerator and denominator, the resulting fraction $= \frac{1}{2}$. Ans. $\frac{1}{3}$.

(41.) Required a fraction such that if its numerator be increased by half its denominator, and its denominator by half its numerator, the resulting fraction $= \frac{3}{4}$. Ans. $\frac{2}{5}$.

(42.) Required a fraction such that if m be added to its numerator it becomes $\frac{a}{b}$, but if n be added to its denominator it becomes $\frac{c}{d}$. Ans. $\frac{bcm + acn}{bdm + bcn}$.

(43.) Required a fraction such that if its numerator be increased by m times its denominator, and its denominator by n times its numerator, the resulting fraction $= \frac{a}{b}$.

Ans. $\frac{a - bm}{b - an}$.

(44.) Find two numbers whose sum $= 40$, and difference $= 12$. Ans. 14 and 26.

(45.) Find two numbers whose sum $= s$, and difference $= d$. Ans. $\frac{1}{2}(s + d)$, $\frac{1}{2}(s - d)$.

(46.) Divide a into three parts such that the first shall be m times as great as the second, and the second m times as great as the third.

Ans. $\frac{m^2 a}{1 + m + m^2}$, $\frac{ma}{1 + m + m^2}$, $\frac{a}{1 + m + m^2}$.

(47.) A farmer has 50 reapers, men and women; the men are to receive 3s. each per day, and the women 2s. 6d.; their wages altogether amount to £7 per day: how many are there of each? Ans. 30 men and 20 women.

(48.) Give a general solution of the last question, supposing the number of reapers to be a, the wages of the men m shillings, and of the women n shillings per day, and the total amount p shillings per day.

Ans. $\frac{p - an}{m - n}$ men, and $\frac{am - p}{m - n}$ women.

(49.) Divide the fraction $\frac{a}{b}$ into two parts, the sum of whose numerators shall be equal to the sum of their denominators. Ans. $\frac{a - b - 1}{b - 1}$, and $\frac{b^2 + b - a}{b(b - 1)}$.

(50.) A certain number of pounds, half-crowns, and shillings together amount to £14, 10s., and the number of half-crowns is double that of the pounds, and half that of the shillings; find the number of each.

Ans. 10 pounds, 20 half-cr., and 40 shillings.

(51.) If a be taken from n times a certain number, the remainder is $= b$; find the number. Ans. $\frac{a + b}{n}$.

(52.) Find a number whose m^{th} and n^{th} parts together $= a$. Ans. $\frac{amn}{m + n}$.

(53.) Find a number the difference of whose m^{th} and n^{th} parts $= a$, m being greater than n. Ans. $\frac{amn}{m - n}$.

(54.) Find a number such that if a be subtracted, and the remainder multiplied by a, the product is the same as when b is subtracted from the number, and the remainder multiplied by b. Ans. $a + b$.

(55.) Divide a into two parts, of which the one shall be the m^{th} part of the other. Ans. $\frac{a}{m + 1}$, and $\frac{am}{m + 1}$.

(56.) Divide p pounds among three persons, giving the second m times, and the third n times as much as the first.

Ans. $\frac{a}{1 + m + n}$, $\frac{am}{1 + m + n}$, $\frac{an}{1 + m + n}$.

(57.) A labourer is engaged for n days, on condition that he receives p pence for every day he works, and pays q pence for every day he is idle. At the end of the time he receives a pence; how many days did he work, and how many was he idle?

Ans. He worked $\frac{nq+a}{p+q}$, and was idle $\frac{np-a}{p+q}$ days.

(58.) At a certain election a persons voted, and the successful candidate had a majority of b votes; how many voted for each? Ans. $\frac{1}{2}(a+b)$, and $\frac{1}{2}(a-b)$.

(59.) If A can perform a piece of work in 8 days, and B in 12 days, how long will they take to finish it if they work both together? Ans. $4\frac{4}{5}$ days.

(60.) If A can perform a piece of work in a days, B in b days, and C in c days, how long will they take working all together? Ans. $\frac{abc}{ab+ac+bc}$.

CHAPTER V.

INVOLUTION.

43. Involution is the raising of a quantity to any required power. This is done by multiplying the quantity continually by itself, till it has been used as factor as often as there are units in the index of the required power.

EXERCISES.

Raise each of the following quantities to the fourth power:

(1.) $4a$. . . . . Ans. $256a^4$.

(2.) $3ax$. . . . . $81a^4x^4$.

(3.) $-a^2b$. . . . . a^8b^4.

(4.) $\frac{a^2x}{by}$. . . . . $\frac{a^8x^4}{b^4y^4}$.

(5.) $\frac{3ax^2y}{7b^2c}$. . . . . $\frac{81a^4x^8y^4}{2401b^8c^4}$.

(6.) $2a-x$. . Ans. $16a^4-32a^3x+24a^2x^2-8ax^3+x^4$.

(7.) $x-\frac{1}{x}$. . . . $x^4-4x^2+6-4x^{-2}+x^{-4}$.

(8.) a^2+x^2. . . . $a^8+4a^6x^2+6a^4x^4+4a^2x^6+x^8$.

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44. By examining the successive developments of a binomial $a+b$; viz.:

2d power, $a^2+2ab+b^2$
3d power, $a^3+3a^2b+3ab^2+b^3$
4th power, $a^4+4a^3b+6a^2b^2+4ab^3+b^4$
5th power, $a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$
&c. &c. &c.;

it will be observed that the powers of a decrease, and those of b increase, by a unit in each successive term, and that the highest power of each is the same as that to which the binomial is raised. Also, if the coefficient of any term be multiplied by the index of a in that term, and the product divided by the number of terms reckoned from the commencement (that is, if it be the third term, divide by 3,—if it be the seventh, by 7, &c. &c.), the result is the coefficient of the next term. If both terms of the binomial be positive, all the terms of the expansion will be positive; but if one term of the binomial be negative, such as b, the terms of the expansion which contain the odd powers of b will be negative.

Thus to raise $a+b$ to the sixth power, the powers of a and b in the successive terms would be a^6, a^5b, a^4b^2, a^3b^3, a^2b^4, ab^5, and b^6. The coefficient of the first term a^6 is 1, and by multiplying the index 6 by 1, and dividing by the number of terms 1, we get 6 for the coefficient of the second term; or as the multiplier and divisor are each unity, it is plain that the coefficient of the second term will always be the same as the index of a in the first. By multiplying 6 the coefficient of the second term by 5, the index of a in that term, and dividing the product by 2, the number of terms, the result is 15, which is the next coefficient. Again, multiplying 15 the coeffieient of the third term by 4, the index of a in that term, and dividing the product by 3, the number of terms, the result is 20, which is the coefficient of the fourth term. Proceeding in the same manner, we would find that

$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6,$$

the signs being all positive. Or in general terms,

$$(a+b)^n = a^n + na^{n-1}b + \frac{n}{1}\,\frac{n-1}{2}a^{n-2}b^2 + - - - - - - - - + \frac{n}{1}\,\frac{n-1}{2}a^2b^{n-2} + \frac{n}{1}ab^{n-1} + b^n.$$

This is the celebrated binomial theorem; and by means of it, a binomial may be raised to any power, without the trouble of the continued multiplication. As this theorem can be shown to hold for all powers, positive or negative, whole or fractional, it is capable of very extensive application.

EXERCISES.

Expand the following by means of the Binomial Theorem.

(1.) $(a+b)^6$.

Ans. $a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$.

(2.) $(a^2-b^2)^4$. . $a^8-4a^6b^2+6a^4b^4-4a^2b^6+b^8$.

(3.) $(a-b)^{\frac{1}{2}}$.

$$a^{\frac{1}{2}}(1-\tfrac{1}{2}a^{-1}b-\frac{1}{2.4}a^{-2}b^2-\frac{1.3}{2.4.6}a^{-3}b^3-\&c.)$$

(4.) $(a+x)^{\frac{3}{2}}$.

$$a^{\frac{3}{2}}(1+\frac{3x}{2a}+\frac{3x^2}{2.4a^2}-\frac{3x^3}{2.4.6a^3}+\frac{3.3x^4}{2.4.6.8a^4}-\&c.)$$

(5.) $\frac{1}{1+x}=(1+x)^{-1}$.

$$1-x+x^2-x^3+x^4-\&c.$$

(6.) $(a^2-b^2)^{-4}$.

$$\frac{.1}{2.3a^8}(1.2.3+2.3.4\frac{b^2}{a^2}+3.4.5\frac{b^4}{a^4}+\&c.)$$

(7.) $(a-x)^{-\frac{1}{4}}$.

$$a^{-\frac{1}{4}}(1+\frac{1.x}{4.a}+\frac{1.5x^2}{4.8a^2}+\frac{1.5.9x^3}{4.8.12a^3}+\&c.)$$

(8.) $2^{\frac{1}{2}}=(1+1)^{\frac{1}{2}}$. . . $1+\frac{1}{2}-\frac{1}{2.4}+\frac{1.3}{2.4.6}-\&c.$

(9.) $2^{\frac{1}{2}} = (4 - 2)^{\frac{1}{2}}$.

$$\text{Ans. } 2 - \tfrac{1}{2} - \tfrac{1}{4}.\ \tfrac{1}{4} - \frac{1.3}{4.6}.\ \tfrac{1}{8} - \frac{1.3.5}{4.6.8}.\ \tfrac{1}{16} - \&c.$$

(10.) $\sqrt{3} = (1 + 2)^{\frac{1}{2}}$.

$$1 + \tfrac{1}{2}.\ 2 - \frac{1}{2.4}.\ 4 + \frac{1.3}{2.4.6}.\ 8 - \&c.$$

EVOLUTION.

45. Evolution is the extraction of any root of a given quantity.

To extract any root of a simple quantity, divide the index of the quantity by the index of the root required.

NOTE 1. Any odd root of a quantity will have the same sign as the quantity itself; but any even root of a positive quantity may be either positive or negative, and no even root of a negative quantity can be extracted.

NOTE 2. Any root of a product may be found by taking that root of each factor; and any root of a fraction by taking that root of both numerator and denominator.

EXERCISES.

Extract the square root of each of the following quantities:

(1.) $9a^2x^2$. . . . . Ans. $\pm 3ax$.

(2.) $100a^6x^8y^{10}$. . . . . $\pm 10a^3x^4y^5$.

(3.) $\dfrac{16a^6y^4}{25b^4x^6}$. . . . . $\pm \dfrac{4a^3y^2}{5b^2x^3}$.

Extract the cube root of each of the following quantities:

(4.) $8a^6b^{12}$. . . . . Ans. $2a^2b^4$.

(5.) $-8a^3b^9$. . . . . $-2ab^3$.

(6.) $-\dfrac{64a^6x^4y^3}{125b^2c^6}$. . . . . $-\dfrac{4a^2x^{\frac{4}{3}}y}{5b^{\frac{2}{3}}c^2}$.

46. To extract the square root of a compound quantity.

(1.) Arrange the terms according to the powers of some one

of the letters, commencing either with the highest or lowest power. (2.) Find the root of the first term, for the first term of the required root; and subtract its square from the given quantity. (3.) For a new divisor, take the double of the part of the root found, and divide the first term of the remainder by it, setting down the quotient for the next term of the required root, and annex this term also to the divisor, then proceed as in common Division; and repeat the same process as often as may be necessary.

EXERCISES.

Extract the square root of each of the following quantities:

(7.) $a^2+2ab+b^2$. . . . . . Ans. $a+b$.

(8.) $4a^2-12ax+9x^2$. . . . . $2a-3x$.

(9.) $x^2-2+\frac{1}{x^2}$. . . . . $x-\frac{1}{x}$.

(10.) $9+24x-68x^2-112x^3+196x^4$. . $3+4x-14x^2$.

47. To extract the cube root of a compound quantity. (1.) Arrange the terms as directed above, and take the cube root of the first term for the first term of the answer. (2.) Subtract the cube of this term from the given quantity, and divide the first term of the remainder by three times the square of the part of the root found; setting down the quotient for the next term of the answer. (3.) Subtract the cube of the binomial thus found from the given quantity; and if there be a remainder, divide its first term by the same divisor as before, for the third term of the answer. (4.) Subtract the cube of the trinomial thus found from the given quantity, and if there still be a remainder, proceed in exactly the same manner as often as may be necessary.

EXERCISES.

Find the cube root of each of the following quantities:

(11.) $x^3+6x^2+12x+8$. . . . Ans. $x+2$.

(12.) $a^3+3a^2b+3ab^2+b^3$. . . . . $a+b$.

(13.) $a^6+6a^5+9a^4-4a^3-9a^2+6a-1$. a^2+2a-1.

(14.) $8x^6-36ax^5+30a^2x^4+45a^3x^3-30a^4x^2-36a^5x-8a^6$.

$2x^2-3ax-2a^2$.

CHAPTER VI.

SURDS OR RADICALS.

48. Any root of a quantity which cannot be exactly determined is called a *Surd* or *Radical*, and sometimes an "*Irrational Quantity.*" Thus $\sqrt{2}$ and $\sqrt[3]{4}$ are surds.

Since the extraction of any root of a product may be performed by extracting that root of each of the factors, many surds may be much simplified by extracting the root of such of the factors as admit of it. Thus $\sqrt{a^4b}=a^2\sqrt{b}$; $\sqrt{108}=\sqrt{36\times 3}=\sqrt{36}\sqrt{3}=6\sqrt{3}$.

EXERCISES.

Simplify the following surds:

(1.) $\sqrt{a^2x}$. . . . . Ans $a\sqrt{x}$.

(2.) $(a^3-a^2b)^{\frac{1}{2}}$. . . . . $a(a-b)^{\frac{1}{2}}$.

(3.) $\left(\dfrac{ax^2-2ax+a}{x^3+2x^2+x}\right)^{\frac{1}{2}}$. . . . . $\dfrac{x-1}{x+1}\left(\dfrac{a}{x}\right)^{\frac{1}{2}}$.

(4.) $\sqrt{75}$. . . . . $5\sqrt{3}$.

(5.) $\sqrt{1728}$. . . . . $24\sqrt{3}$.

(6.) $\sqrt{245}$. . . . . $7\sqrt{5}$.

49. Any quantity may be reduced to the form of a surd by raising it to any power, and then indicating the corresponding root. Thus, $a=\sqrt{a^2}=\sqrt[3]{a^3}$, &c. &c. $2\sqrt{5}=\sqrt{4}\sqrt{5}=\sqrt{20}$. $2\sqrt[3]{3}=\sqrt[3]{8}\sqrt[3]{3}=\sqrt[3]{24}$.

EXERCISES.

Reduce each of the following quantities to the form of the square root, cube root, and n^{th} root:

(7.) $2a$. . . Ans. $\sqrt{4a^2}$, $\sqrt[3]{8a^3}$, $\sqrt[n]{2^na^n}$.

(8.) $2a^{\frac{1}{2}}$. . . . . $\sqrt{4a}$, $\sqrt[3]{8a^{\frac{3}{2}}}$, $\sqrt[n]{2^na^{\frac{n}{2}}}$.

(9.) $a-b$. . . . $\sqrt{(a-b)^2}$, $\sqrt[3]{(a-b)^3}$, $\sqrt[n]{(a-b)^n}$.

(10.) $\frac{1}{2}\sqrt{6x}$. . . . $\sqrt{\frac{3x}{2}}$, $\sqrt[5]{\left(\frac{3x}{2}\right)^{\frac{5}{2}}}$, $\sqrt[n]{\left(\frac{3x}{2}\right)^{\frac{n}{2}}}$.

(11.) $3\sqrt[3]{5}$. . . . . $\sqrt{(135)^{\frac{2}{3}}}$, $\sqrt[3]{135}$, $\sqrt[n]{(135)^{\frac{n}{3}}}$.

(12.) $\frac{1}{2}\sqrt[3]{4}$. . . . $\sqrt{(4)^{-\frac{1}{3}}}$, $\sqrt[3]{\frac{1}{2}}$, $\sqrt[n]{(\frac{1}{2})^{\frac{n}{3}}}$.

50. Quantities having different indices may be reduced to equivalent quantities that shall have the same index, by reducing the indices to a common denominator, then raising each of the quantities to the power denoted by the numerator of its index, and indicating the root denoted by the common denominator. Thus, $a^{\frac{1}{2}} + b^{\frac{1}{3}} = a^{\frac{3}{6}} + b^{\frac{2}{6}} = (a^3)^{\frac{1}{6}} + (b^2)^{\frac{1}{6}}$. $2^{\frac{1}{2}} + 3^{\frac{1}{3}} = 2^{\frac{3}{6}} + 3^{\frac{2}{6}} = (2^3)^{\frac{1}{6}} + (3^2)^{\frac{1}{6}} = 8^{\frac{1}{6}} + 9^{\frac{1}{6}}$.

EXERCISES.

Reduce the following quantities to others having a common index:

(13.) $2^{\frac{1}{3}}$ and $3^{\frac{1}{2}}$. . . . . Ans. $4^{\frac{1}{6}}$ and $27^{\frac{1}{6}}$.

(14.) a^2 and $a^{\frac{1}{2}}$. . . . . . . $(a^4)^{\frac{1}{2}}$ and $a^{\frac{1}{2}}$.

(15.) $a^{\frac{1}{4}}$ and $b^{\frac{1}{3}}$. . . . . . . $(a^3)^{\frac{1}{12}}$ and $(b^4)^{\frac{1}{12}}$.

(16.) $2^{\frac{1}{3}}$ and $5^{\frac{1}{2}}$. . . . . . . $4^{\frac{1}{6}}$ and $(125)^{\frac{1}{6}}$.

ADDITION AND SUBTRACTION OF SURDS.

51. To Add or Subtract Surds; Reduce them if necessary to their simplest form, and proceed as with rational quantities.

EXERCISES.

(1.) $\sqrt{18}+\sqrt{200}+\sqrt{32}$. . . . . Ans. $17\sqrt{2}$.

(2.) $2\sqrt{18}-3\sqrt{8}+2\sqrt{50}$. . . . . . $10\sqrt{2}$.

(3.) $\frac{3}{2}\sqrt{\frac{5}{9}}+\frac{2}{3}\sqrt{\frac{5}{4}}-\frac{5}{4}\sqrt{80}$. . . . . . $-2\frac{1}{6}\sqrt{5}$.

(4.) $\sqrt[3]{32}+\sqrt[3]{108}-\sqrt[3]{256}$. . . . . . $\sqrt[3]{4}$.

(5.) $\sqrt{a^2x}-\sqrt{b^2x}+\sqrt{x^3}$. . . . . . $(a-b+x)x^{\frac{1}{2}}$.

(6.) $\sqrt[3]{32a^3x} - \sqrt[3]{108x^4}$. . . . . Ans. $(2a+3x)\sqrt[3]{4x}$.

(7.) $(a^3+2a^2x+ax^2)^{\frac{1}{2}}+(a^3-2a^2x+ax^2)^{\frac{1}{2}}$. $2a^{\frac{3}{2}}$.

(8.) $\sqrt[4]{243x} - \sqrt[4]{768x} + \sqrt[4]{48x}$. . . . . $\sqrt[4]{3x}$.

MULTIPLICATION AND DIVISION OF SURDS.

52. The multiplication and division of surds are conducted on the same principles as the multiplication and division of rational quantities. Thus $\sqrt{a} \times \sqrt{b} = a^{\frac{1}{2}}b^{\frac{1}{2}} = (ab)^{\frac{1}{2}}$; $2^{\frac{1}{2}} \times 3^{\frac{1}{2}} = 6^{\frac{1}{2}}$; $x^{\frac{1}{2}} \times x^{\frac{1}{3}} = x^{\frac{5}{6}}$; $2^{\frac{1}{2}} \times 3^{\frac{1}{3}} = 8^{\frac{1}{6}} \times 9^{\frac{1}{6}} = (72)^{\frac{1}{6}}$.

EXERCISES.

Surds to be multiplied:

(1.) $\sqrt{2} \times \sqrt{5}$. . . . . Ans. $\sqrt{10}$.

(2.) $2\sqrt{3} \times 3\sqrt{5}$. . . . . $6\sqrt{15}$.

(3.) $2\sqrt{a} \times 3\sqrt{b} \times 2\sqrt{c}$. . . . . $12\sqrt{abc}$.

(4.) $2^{\frac{1}{2}} \times 3^{\frac{1}{3}} \times 4^{\frac{1}{4}}$. . . . . $2\sqrt[3]{3}$.

(5.) $\sqrt{12x} \times \sqrt{6x}$. . . . . $6x\sqrt{2}$.

(6.) $(\sqrt{3}-1) \times (\sqrt{3}+1)$. . . . . 2.

(7.) $(x^{\frac{3}{2}} - 2x + x^{\frac{1}{2}} - 1)(x^{\frac{1}{2}}+1)$. . . . $x^2 - x^{\frac{3}{2}} - x - 1$.

(8.) $2a^{\frac{1}{2}} \times 3a^{\frac{1}{3}} \times 4a^{\frac{1}{4}}$. . . . . $24a^{\frac{13}{12}}$.

(9.) $(9+\sqrt{17})^{\frac{1}{3}} + (9-\sqrt{17})^{\frac{1}{3}}$. . . . . . 4.

(10.) $\{(a+x)^{\frac{1}{2}} + (a-x)^{\frac{1}{2}}\} \times \{(a+x)^{\frac{1}{2}} - (a-x)^{\frac{1}{2}}\}$. $2x$.

(11.) $(10+\sqrt{19})^{\frac{1}{4}} \times (10-\sqrt{19})^{\frac{1}{4}}$. . . . . 3.

(12.) $(\sqrt{a}+\sqrt{b}+\sqrt{c}) \times (-\sqrt{a}+\sqrt{b}+\sqrt{c}) \times (\sqrt{a} - \sqrt{b}+\sqrt{c}) \times (\sqrt{a}+\sqrt{b}-\sqrt{c})$.
$-a^2-b^2-c^2+2(ab+ac+bc)$.

Surds to be divided:

(13.) $\sqrt{20} \div \sqrt{5}$. . . . . Ans. 2.

(14.) $4\sqrt[3]{81} \div 3\sqrt[3]{3}$. . . . . 4.

(15.) $4\sqrt{a} \div 2\sqrt[3]{a}$. . . . . . . Ans. $2a^{\frac{1}{6}}$.

(16.) $(a^2 - x^2)^{\frac{1}{2}} \div (a-x)$. . . . . $\sqrt{\dfrac{a+x}{a-x}}$.

(17.) $(\sqrt{20} - \sqrt{12}) \div (\sqrt{5} - \sqrt{3})$. . . 2.

(18.) $\left(x+1+\frac{1}{x}\right) \div \left(x^{\frac{1}{2}} - 1 + \frac{1}{x^{\frac{1}{2}}}\right)$. . . $x^{\frac{1}{2}} + 1 + x^{-\frac{1}{2}}$.

(19.) $\left\{(1-x)^{\frac{1}{2}} + \frac{1}{(1+x)^{\frac{1}{2}}}\right\} \div \left\{1 + \frac{1}{(1-x^2)^{\frac{1}{2}}}\right\}$. $(1-x)^{\frac{1}{2}}$.

(20.) $(\sqrt[4]{a^3} - \sqrt[4]{b^3}) \div (\sqrt[4]{a} - \sqrt[4]{b})$. . . $a^{\frac{1}{2}} + (ab)^{\frac{1}{4}} + b^{\frac{1}{2}}$.

---

53. The square root of a quantity cannot be partly rational and partly irrational. For if possible let $\sqrt{x} = a + \sqrt{b}$, then by squaring these equals, $x = a^2 + 2a\sqrt{b} + b$, and hence $\sqrt{b} = \frac{x - a^2 - b^2}{2a}$, which is impossible.

54. From this it follows that if an equation be of the form $x + \sqrt{y} = a + b + \sqrt{c}$, the rational parts must be equal, and also the irrational parts, viz, $x = a + b$, and $\sqrt{y} = \sqrt{c}$.

55. If a binomial surd be of the form $a + \sqrt{b}$, one term being rational, the square root may be extracted thus:

Assume $(a+\sqrt{b})^{\frac{1}{2}} = \sqrt{x} + \sqrt{y}$

then by squaring, $a + \sqrt{b} = x + y + 2\sqrt{xy}$

and hence, (54.) $x + y = a$, and $2\sqrt{xy} = \sqrt{b}$.

Squaring each of these equals,

we get $x^2 + 2xy + y^2 = a^2$; and $4xy = b$

* $\therefore x^2 - 2xy + y^2 = a^2 - b$; and $x - y = \sqrt{a^2 - b}$.

But $x + y = a$;

hence $x = \dfrac{a + \sqrt{a^2 - b}}{2}$; and $y = \dfrac{a - \sqrt{(a^2 - b)}}{2}$

$\therefore \sqrt{x} + \sqrt{y}$ or $(a + \sqrt{b})^{\frac{1}{2}} = \left(\dfrac{a + \sqrt{a^2 - b}}{2}\right)^{\frac{1}{2}} + \left(\dfrac{a - \sqrt{a^2 - b}}{2}\right)^{\frac{1}{2}}$.

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* The sign $\therefore$ is sometimes used as an abbreviation for "Therefore."

It thus appears that when $a^2 - b$ is a perfect square, the square root of $a + \sqrt{b}$ may be expressed by a binomial, which may be found by assuming it $= \sqrt{x} + \sqrt{y}$, and proceeding as above, or by substitution in the general formula.

If the surd be of the form $a - \sqrt{b}$, assume the root $= \sqrt{x} - \sqrt{y}$, and proceed as before.

If the surd be of the form $\sqrt{a^2c} \pm \sqrt{bc}$, or can be put under the form $\sqrt{c}(a \pm \sqrt{b})$, its square root may be found by first finding the root of $a \pm \sqrt{b}$, and then multiplying the resnlt by $\sqrt[4]{c}$.

EXERCISES.

Extract the square root of the following surds:

(1.) $4 + 2\sqrt{3}$. . . . . Ans. $1 + \sqrt{3}$.

(2.) $9 + 4\sqrt{5}$. . . . . $2 + \sqrt{5}$.

(3.) $11 - 6\sqrt{2}$. . . . . $3 - \sqrt{2}$.

(4.) $30 - 12\sqrt{6}$. . . . . $3\sqrt{2} - 2\sqrt{3}$.

(5.) $\sqrt{27} + \sqrt{24}$. . . . . $\sqrt[4]{3} + \sqrt[4]{12}$.

(6.) $3\sqrt{2} + 4 = 3\sqrt{2} + \sqrt{16}$. . . . $\sqrt[4]{2} + \sqrt[4]{8}$.

(7.) $x + 2(x-1)^{\frac{1}{2}}$. . . . . $1 + (x-1)^{\frac{1}{2}}$.

(8.) $2a - 2(2ax - x^2)^{\frac{1}{2}}$. . . . . $x^{\frac{1}{2}} - (2a - x)^{\frac{1}{2}}$.

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56. A fractional surd may in most cases be much simplified by multiplying numerator and denominator by such a quantity as will render the denominator rational. Thus, if the denominator be of the form $b^{\frac{m}{n}}$, the multiplier is $b^{1-\frac{m}{n}}$; if it be of the form $a^{\frac{1}{2}} \pm b^{\frac{1}{2}}$, the multiplier is $a^{\frac{1}{2}} \mp b^{\frac{1}{2}}$; if it be of the form $a^{\frac{1}{3}} \pm b^{\frac{1}{3}}$, the multiplier is $a^{\frac{2}{3}} \mp a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}$; and if it be of the form $a^{\frac{1}{4}} \pm b^{\frac{1}{4}}$, first multiply by $a^{\frac{1}{4}} \mp b^{\frac{1}{4}}$, then the result $a^{\frac{1}{2}} - b^{\frac{1}{2}}$ multiplied by $a^{\frac{1}{2}} + b^{\frac{1}{2}}$ will be rational.

A general formula for all such multipliers may easily be proved, but it is seldom of use.

EXERCISES.

Express each of the following fractions with a rational denominator:

(1.) $\frac{2}{\sqrt{3}}$. . . . . Ans. $\frac{2\sqrt{3}}{3}$.

(2.) $\frac{3\sqrt{a}}{2\sqrt{b}}$. . . . . $\frac{3\sqrt{ab}}{2b}$.

(3.) $\frac{2a}{3^{\frac{1}{3}}}$. . . . . $\frac{2.3^{\frac{2}{3}}a}{3} = \frac{2.9^{\frac{1}{3}}a}{3}$.

(4.) $\frac{3ax}{3^{\frac{1}{4}}}$. . . . . $27^{\frac{1}{4}}ax$.

(5.) $\frac{2+\sqrt{3}}{2-\sqrt{3}}$. . . . . $7+4\sqrt{3}$.

(6.) $\frac{3}{\sqrt{3}+\sqrt{2}}$. . . . . $3(\sqrt{3}-\sqrt{2})$.

(7.) $\frac{3\sqrt{2}-2\sqrt{3}}{3\sqrt{2}+2\sqrt{3}}$. . . . . $5-2\sqrt{6}$.

(8.) $\frac{(a+x)^{\frac{1}{2}}-(a-x)^{\frac{1}{2}}}{(a+x)^{\frac{1}{2}}+(a-x)^{\frac{1}{2}}}$. . . . . . $\frac{a-(a^2-x^2)^{\frac{1}{2}}}{x}$.

(9.) $\frac{1}{3^{\frac{1}{3}}-2^{\frac{1}{3}}}$. . . . . $9^{\frac{1}{3}}+6^{\frac{1}{3}}+4^{\frac{1}{3}}$.

(10.) $\frac{1}{3^{\frac{1}{4}}+2^{\frac{1}{4}}}$. . . . . $27^{\frac{1}{4}}+12^{\frac{1}{4}}-18^{\frac{1}{4}}-8^{\frac{1}{4}}$.

CHAPTER VII.

EQUATIONS CONTAINING SURDS.

57. When an equation contains a surd, if, by transposition or other operations, the surd be made to stand alone on one side of the equation, and both members be then raised to the power corresponding to the root expressed in the surd, an equation will be obtained which will be free of surds.

If the equation contain more than one surd, the same process must be repeated as often as may be necessary.

EXERCISES.

Resolve the following equations:

(1.) $\sqrt{x} = 3$. . . . . Ans. $x = 9$.

(2.) $2 + \sqrt{x} = 4$. . . . . $x = 4$.

(3.) $\sqrt{(x+5)} + 2 = 5$. . . . . $x = 4$.

(4.) $x + \sqrt{(x^2+5)} = 5$. . . . . . . $x = 2$.

(5.) $3 + \sqrt{(x+9)} = x$. . . . . $x = 7$.

(6.) $x + \sqrt{(x^2+a^2)} = b$. . . . . . $x = \frac{b^2 - a^2}{2b}$.

(7.) $x - \sqrt{(x^2-a^2)} = b$. . . . . . $x = \frac{a^2 + b^2}{2b}$.

(8.) $x + a = \sqrt{(x^2+b^2)}$. . . . . . $x = \frac{b^2 - a^2}{2a}$.

(9.) $2x + \sqrt{(3x^2+81)} + 8 = 5x - 1$. $x = 9$.

(10.) $2x + b = \sqrt{(x^2 - a^2)} + x$. . . . $x = -\frac{a^2+b^2}{2b}$.

(11.) $6x + 1 = 9x - \sqrt{(9x^2-5)}$. . . . $x = 1$.

(12.) $\sqrt{(a^2 + b^2x)} - c + dx = (b+d)x$. . $x = \frac{a^2 - c^2}{2bc}$.

(13.) $\sqrt{(x+5)} = \sqrt{x} + 5$. . . . . . $x = 4$.

(14.) $\sqrt{(x+a)} = \sqrt{x} + a$. . . . . . $x = \left(\frac{1-a}{2}\right)^2$.

(15.) $\sqrt{(x-9)} + \sqrt{(x-12)} = 3$. . . $x = 13$.

(16.) $\sqrt{(x^2-9)} + 2 = \sqrt{(x^2+11)}$. . . $x = \pm 5$.

(17.) $3\sqrt{x} + 4\sqrt{x} - 5\sqrt{x} = 6$. . . . $x = 9$.

(18.) $(7+x)^{\frac{1}{2}} = 7 - x^{\frac{1}{2}}$. . . . . . $x = 9$.

(19.) $\sqrt{(2x-3a)} = \sqrt{2x} - 3\sqrt{a}$. . . $x = 2a$.

(20.) $\sqrt{x} + \sqrt{(x-5)} = 10(x-5)^{-\frac{1}{2}}$. $x = 9$.

(21.) $(7+x)^{\frac{1}{2}} - x^{\frac{1}{2}} = (3+x)^{\frac{1}{2}} + x^{\frac{1}{2}}$. . $x = \frac{1}{3}$.

(22.) $(2x-45)^{\frac{1}{2}} = 3\sqrt{15} - \sqrt{2x}$. . . $x = 30$.

(23.) $\sqrt{x} + \sqrt{(x+a)} = 2a(x+a)^{-\frac{1}{2}}$. . $x = \frac{1}{3}a$.

(24.) $(x+4)^{\frac{1}{2}} - (x-3)^{\frac{1}{2}} = 4(x+4)^{-\frac{1}{2}}$. $x = 12$.

(25.)* $\dfrac{(x+1)^{\frac{1}{2}}-1}{(x+1)^{\frac{1}{2}}+1} = \frac{1}{2}$. . . . . Ans. $x = 8$.

(26.) $\dfrac{\sqrt{3x}+\sqrt{(10-3x)}}{\sqrt{3x}-\sqrt{(10-3x)}} = 2$. . . . . . . $x = 3$.

(27.) $\dfrac{6-\sqrt{(x-2)}}{6+\sqrt{(x-2)}} = \frac{1}{5}$. . . . . $x = 18$.

(28.) $\dfrac{a+\sqrt{x}}{a-\sqrt{x}} = \dfrac{b+3\sqrt{x}}{b+2\sqrt{x}}$. . . . . . $x = \left(\dfrac{a-2b}{5}\right)^2$.

(29.) $(8+x)^{\frac{1}{2}} = 2(1+x)^{\frac{1}{2}} + x^{\frac{1}{2}}$. . . . $x = \frac{1}{3}$.

(30.) $\sqrt{x} + \sqrt{(x - \sqrt{1-x})} = 1$. . . $x = \frac{16}{25}$.

(31.) $(1+x)^{\frac{1}{3}} + (1-x)^{\frac{1}{3}} = 2^{\frac{1}{3}}$. . . . $x = 1$.

(32.) $(a+x)^{\frac{1}{3}} + (a-x)^{\frac{1}{3}} = b$. $x = \sqrt{a^2 - \left(\dfrac{b^3-2a}{3b}\right)^3}$.

(33.) $\dfrac{(x+1)^{\frac{1}{2}}+(x-1)^{\frac{1}{2}}}{(x+1)^{\frac{1}{2}}-(x-1)^{\frac{1}{2}}} = 2$. . . . . $x = \frac{5}{4}$.

(34.) $\dfrac{(x+a)^{\frac{1}{2}}+(x-a)^{\frac{1}{2}}}{(x+a)^{\frac{1}{2}}-(x-a)^{\frac{1}{2}}} = a$. . . . . $x = \frac{1}{2}(a^2+1)$.

(35.) $(x+1)^{\frac{1}{2}} - (x-1)^{\frac{1}{2}} = (x^2-1)^{\frac{1}{4}}$. . $x = \frac{3}{5}\sqrt{5}$.

(36.) $(a+x)^{\frac{1}{2}} + (a-x)^{\frac{1}{2}} = (a^2-x^2)^{\frac{1}{4}}$. $x = a\sqrt{-3}$.

(37.) $(a+x)^{\frac{1}{4}} - (a-x)^{\frac{1}{4}} = (a^2-x^2)^{\frac{1}{8}}$. $x = \frac{3}{7}a\sqrt{5}$.

(38.) $(x^2+x)^{\frac{1}{2}} + (x^2-x)^{\frac{1}{2}} = 2(x^2-1)^{\frac{1}{2}}$. $x = \frac{2}{3}\sqrt{3}$.

(39.) $(x^2+2ax)^{\frac{1}{2}} + (x^2-2ax)^{\frac{1}{2}} = \dfrac{x^2}{(x^2+2ax)^{\frac{1}{2}}}$. $x = a\sqrt{8}$.

(40.) $a(1-x^2)^{\frac{1}{2}} - x(1-a^2)^{\frac{1}{2}} = (a^2-x^2)^{\frac{1}{2}}$. $x = a$.

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* When an equation consists of two equal fractions, it may often be simplified by putting the sum of the numerator and denominator of the one fraction, divided by their difference, equal to the sum of the numerator and denominator of the other, divided by their difference.

(41.) $\dfrac{(x+1)^{\frac{1}{3}}-(x-1)^{\frac{1}{3}}}{(x+1)^{\frac{1}{3}}+(x-1)^{\frac{1}{3}}}=\frac{1}{3}$. . . . Ans. $x=1\frac{2}{7}$.

(42.) $\dfrac{a+x+\sqrt{(2ax+x^2)}}{a+x-\sqrt{(2ax+x^2)}}=b^2$. . . . $x=\frac{a}{2b}(b-1)^2$.

(43.) $x-y=a^2$, and $x^{\frac{1}{2}}-y^{\frac{1}{2}}=b$.

$$x=\left(\frac{a^2+b^2}{2b}\right)^2,\ y=\left(\frac{a^2-b^2}{2b}\right)^2.$$

(44). $2x+\sqrt{y}=12$, and $4x^2-y=96$. . $x=5,\ y=4$.

(45.) $x(y^2+z^2)^{\frac{1}{2}}=\sqrt{13}$. . . . . . $x=1$.

$y(x^2+z^2)^{\frac{1}{2}}=2\sqrt{10}$. . . . . . $y=2$.

$z(x^2+y^2)^{\frac{1}{2}}=3\sqrt{5}$. . . . . . $z=3$.

CHAPTER VIII.

QUADRATIC EQUATIONS.

58. A *Quadratic Equation*, or an equation of the second degree, is one which contains the second power of the unknown quantity. An equation containing the second power *only* is called a pure quadratic, but one containing both the first and second powers is called a compound quadratic.

The resolution of pure quadratics, or of equations of any degree, containing only one power of the unknown quantity, is effected in precisely the same manner as in simple equations, except that at the conclusion the root corresponding to the degree of the equation is to be extracted.

All compound quadratic equations may be reduced to the form

$$ax^2+bx=c,$$

in which x is the unknown quantity; a, b, and c, known numbers; a being positive, but b and c either positive or negative.

To resolve this general equation,

Multiply by $4a$, $\quad 4a^2x^2 + 4abx = 4ac$

Add b^2, $\quad 4a^2x^2 + 4abx + b^2 = b^2 + 4ac$

Extract the square root, $\quad 2ax + b = \pm\sqrt{(b^2 + 4ac)}$.

Transpose b, and divide by $2a$, $\quad x = \frac{-b + \sqrt{(b^2 + 4ac)}}{2a}$.

Hence, to resolve an equation of the second degree, reduce it if necessary to the form $ax^2 + bx = c$; then to find x, to the coefficient of the lower power, with its sign changed, annex by the signs $\pm$ the square root of the quantity obtained, by adding the square of that coefficient to four times the product of the coefficient of the higher power and the second member; and, lastly, divide the whole expression thus found by twice the coefficient of the higher power.

This is merely expressing the general formula, $x = \frac{-b \pm \sqrt{(b^2 + 4ac)}}{2a}$, in English; and in all particular examples the value of the unknown quantity may be found either by following the above directions, or by simply substituting the values of a, b, and c, in this general expression. Thus,

Let $6x^2 - 11x = 10$ be the given equation. Comparing this with the general equation $ax^2 + bx = c$, we find that $a = 6$, $b = -11$, and $c = 10$. Hence, by substitution,

$$x = \frac{11 \pm \sqrt{(121 + 240)}}{12},$$

$$\text{or} \quad x = \frac{11 \pm 19}{12}.$$

$$\therefore \quad x = 2\tfrac{1}{2} \text{ or } -\tfrac{2}{3}.$$

There are therefore *two* values for x; and that this is always the case appears from the general formula, which will give one value for x when the surd is taken positive, and another when it is taken negative. But as it admits of no other variety, there can be *only two* values.

Also, if c be negative, and $4ac$ greater than b^2, the expression $\sqrt{(b^2 + 4ac)}$ becomes the square root of a negative quantity, which is impossible. In this case, therefore, there are no values of x which will satisfy the required conditions.

59. When the coefficient of the lower power is an even number, the general equation may be put under the form

$ax^2 + 2bx = c$, where b denotes half the coefficient.

Multiplying by a, $\quad a^2x^2 + 2abx = ac.$

Adding b^2, $\quad a^2x^2 + 2abx + b^2 = b^2 + ac.$

Extracting the square root, $\quad ax + b = \pm\sqrt{(b^2 + ac)}.$

Transposing b, and dividing by a, $\quad x = \dfrac{-b \pm \sqrt{(b^2+ac)}}{a}.$

By means of this formula the values of x in this case may be obtained rather more easily than by the other, and it may be expressed in English as follows:

To find x; to half the coefficient of the lower power annex, by the signs $\pm$, the square root of the quantity obtained by adding the square of this half-coefficient to the product of the coefficient of the higher power, and the second member; and, lastly, divide the whole by the coefficient of the higher power.

It may be observed that all quadratics may be resolved by either formula, the latter being preferable when the coefficient of the lower power is an even number.

NOTE. In addition to the general rules exemplified in (40.) and (41.), various expedients are often employed, by means of which equations may be simplified, and much more easy and elegant solutions obtained. A thorough knowledge of these, however, with skill and dexterity in applying them, can only be acquired by extensive practice. The following are some of the more generally useful:—

(1.) An equation may often be simplified by the addition or subtraction of *unity* on both sides, or by subtracting both sides from *unity*.

(2.) By transposing all the terms to one side, resolving the expression thus found into two or more factors, and putting each factor $= 0$.

(3.) When there are two unknown quantities equally involved in both equations, the sum and difference of two other unknown quantities may be substituted for them.

(4.) When the sum of the dimensions of the unknown quantities is the same in every term of both equations, we may substitute for one of the unknown quantities the product of the other and a third.

EXERCISES.

(1.) $\dfrac{x^2}{2} + \dfrac{x^2}{3} = \dfrac{x^2}{4} + 2\frac{1}{3}$. . . . . . Ans. $x = \pm 2$.

(2.) $3x^2 - \frac{x^2}{2} = 54 - 2x^2$. Ans. $= \pm\sqrt{12} = \pm 2\sqrt{3}$.

(3.) $(x+1)^2 = 2(x+5)$. . . . . $x = \pm 3$.

(4.) $(x+3)^2 = 6x + 2x^2$. . . . . $x = \pm 3$.

(5.) $\frac{5}{1+x} + \frac{5}{1-x} = -2$. . . . . $x = \pm\sqrt{6}$.

(6.) $3x^2 - \frac{4x^2-1}{5} = x^2 + \frac{3x^2-2}{2}$. . $x = \pm 2$.

(7.) $x^2 = 4x - 3$. . . . . $x = 3$, or 1.

(8.) $x^2 = 6 - x$. . . . . $x = 2$, or -3.

(9.) $x^2 - 5x = 10 - 2x$. . . . . . $x = 5$, or -2.

(10.) $x^2 + 8x = 2x - 8$. . . . . . $x = -2$, or -4.

(11.) $6x - x^2 = 5$. . . . . $x = 5$, or 1.

(12.) $6x^2 = 5x - 1$. . . . . $x = \frac{1}{2}$, or $\frac{1}{3}$.

(13.) $x^2 + \frac{7x}{2} = 2$. . . . . $x = \frac{1}{2}$, or -4.

(14.) $15x - 8x^2 = 4\frac{1}{2}$. . . . . $x = \frac{3}{2}$, or $\frac{3}{8}$.

(15.) $\frac{x^2}{2} + \frac{2x}{3} = \frac{2}{3}$. . . . . $x = \frac{2}{3}$, or -2.

(16.) $x^2 - 5x + 6 = 0$. . . . . $x = 3$, or 2.

(17.) $\frac{1}{3}x^2 - \frac{1}{2}x - \frac{5}{6} = 0$. . . . . . $x = \frac{5}{2}$, or -1.

(18.) $11x^2 + 6 = \frac{41x}{2}$. . . . . . . $x = \frac{3}{2}$, or $\frac{4}{11}$.

(19.) $80x^2 + 18x + 1 = 0$. . . . . . $x = -\frac{1}{8}$, or $-\frac{1}{10}$.

(20.) $\frac{2}{3}x^2 - \frac{3}{4}x = \frac{3}{8}$. . . . . $x = \frac{3}{2}$, or $-\frac{3}{8}$.

(21.) $2x^4 - 3x^2 = 20$. . . . . $x = \pm 2$, &c.

(22.) $x + 2\sqrt{x} = 15$. . . . . $x = 25$, or 9.

(23.) $x^6 - 30x^3 = -81$. . . . . . . $x = 3$, $\sqrt[3]{3}$, &c.

(24.) $x^{\frac{1}{2}} - 2x^{-\frac{1}{2}} = 1$. . . . . $x = 4$, or 1.

(25.) $5(x^2 - 1) - 4(3x - 2) + 1 = 0$. $x = 2$, or $\frac{2}{5}$.

(26.) $4(x-2)^2 - 1 = 5(x+2)$. . . $x = 5$, or $\frac{1}{4}$.

(27.) $\frac{3}{8}(x^2 - 4) = \frac{3}{2}(x - 1)$. . . . . . $x = 4$, or 0.

(28.) $\frac{x+1}{x-1} - \frac{x-1}{x+1} = 1$. . . . . . $x = 2 \pm \sqrt{5}$.

(29.) $\frac{x-2}{x-3}+\frac{x-1}{x-2}=2\frac{5}{6}$. . . . Ans. $x=5$, or $2\frac{2}{5}$.

(30.) $\frac{2}{2-x}+\frac{4}{3-x}=4$. . . . . . $x=2\frac{1}{2}$, or 1.

(31.) $\frac{x}{x-1}-\frac{x-1}{x}=\frac{3}{2}$. . . . . . $x=2$, or $\frac{1}{3}$.

(32.) $\frac{3x-1}{3x+1}-\frac{1}{21}=\frac{5x-2}{5x+2}$. . . . . $x=2$, or $\frac{1}{15}$.

(33.) $3(2-x)+2(3-x)=2(4+3x^2)$. $x=\frac{1}{2}$, or $-1\frac{1}{3}$.

(34.) $x^2+1+(x+1)^2=(x+2)(x+3)$. $x=4$, or -1.

(35.) $\frac{x}{2x+3}=\frac{2}{3(x-1)}$. . . . . . $x=3$, or $-\frac{2}{3}$.

(36.) $\frac{1}{3+x}-\frac{1}{3+2x}=\frac{1}{18}$. . . . . $x=3$, or $\frac{3}{2}$.

(37.) $(x+3)^2-12=x+3$. . . . $x=1$, or -6.

(38.) $5x^2-5(6-x)^2=24x(6-x)$. . $x=5$, or $-\frac{3}{2}$.

(39.) $\frac{2x+3}{x}-\frac{3x-4\frac{1}{2}}{5x+2\frac{1}{2}}=3$. . . . $x=2\frac{1}{2}$, or $-\frac{3}{8}$.

(40.) $\frac{2x+3}{2x-3}-\frac{16}{15}=\frac{2x-3}{2x+3}$. . . . $x=6$, or $-\frac{3}{8}$.

(41.) $x+\sqrt{4x+1}=11$. . . . . . $x=6$, or 20.

(42.) $3x-2\sqrt{(x-2)}=27$. . . . . $x=11$, or $7\frac{4}{9}$.

(43.) $(x+3)^{\frac{1}{2}}+(x-3)^{\frac{1}{2}}=(3x+3)^{\frac{1}{2}}$. $x=5$, or -3.

(44.) $2\sqrt{(3x-5)}=3+\sqrt{(5x-1)}$. . $x=10$, or $\frac{82}{49}$.

(45.) $\sqrt{(x-1)(x-2)}+\sqrt{(x-3)(x-4)}=\sqrt{2}$. $x=2$, or 3.

(46.) $x^2-\sqrt{(x^2-7)}=13$. . . . . $x=\pm 4$, &c.

(47.) $x^2-x-3\sqrt{(3x^2-4x+1)}=\frac{x}{3}-7$. $x=3$, or $-\frac{5}{3}$, &c.

(48.) $5x-3\sqrt{x}=14$. . . . . $x=4$, or $\frac{49}{25}$.

(49.) $x^2=31+\sqrt{(x^2-11)}$. . . . $x=\pm 6$, or $\pm 3\sqrt{3}$.

(50.) $2x-\sqrt{(2x+6)}=6$. . . . . $x=5$, or $\frac{3}{2}$.

(51.) $3x^2+2\sqrt{(2x^2-3x+7)}=x^2+3x+17$. $x=3$, or $-\frac{3}{2}$, &c.

(52.) $9x-4x^2+\sqrt{(4x^2-9x+11)}=5$. $x=2$, or $\frac{1}{4}$, &c.

Ans.

(53.) $3x(3-x)=11-4\sqrt{(x^2-3x+5)}$. $x=\frac{1}{2}(3\pm\sqrt{5})$, &c.

(54.) $x^3-5x^{-3}=4$. . . . . $x=\sqrt[3]{5}, -1$, &c.

(55.) $3(x+5)^{\frac{1}{3}}-2(x+5)^{-\frac{1}{3}}=5$. . $x=3$, or $-5\frac{1}{27}$.

(56.) $x^n+ax^{-n}=b$. . . . . $x=\left(\frac{b+\sqrt{(b^2-4a)}}{2}\right)^{\frac{1}{n}}$.

(57.) $\frac{2x(a-x)}{3a-2x}=4a$. . . . . $x=3a$, or $2a$.

(58.) $\sqrt{(a+x)}+\sqrt{(a-x)}=\frac{12a}{5\sqrt{(a+x)}}$. $x=\frac{4}{5}a$, or $\frac{3}{5}a$.

(59.) $\frac{a+x}{\sqrt{(a-x)}}+\frac{a-x}{\sqrt{(a+x)}}=2\sqrt{a}$. $x=\pm a(\pm 8\sqrt{2}-11)^{\frac{1}{2}}$.

(60.) $x^2+2a\sqrt{(x^2+ax-a^2)}=4a^2-ax$. $x=a$, or $-2a$, &c.

(61.) $2(x^2-ax)-\sqrt{(x^2-ax-a^2)}=a(4a-1)$.

$x=2a$, or $-a$, &c.

(62.) $\left(\frac{x}{x-1}\right)^2+\left(\frac{x}{x+1}\right)^2=n(n-1)$.

$x=\pm\sqrt{\frac{n}{n-2}}$, or $\pm\sqrt{\frac{n-1}{n+1}}$.

(63.) $x^2+y^2=34$, and $xy=15$. $x=\pm 5$; $y=\pm 3$.

(64.) $x^2-y^2=27$, and $xy-y^2=9$. $x=\pm 6$; $y=\pm 3$.

(65.) $x^2+y^2=20$, and $x+y=6$. $x=4$, or 2; $y=2$, or 4.

(66.) $x+y=a$. . . . . . . $x=\frac{1}{2}a\pm\frac{1}{2}(a^2-b^2)^{\frac{1}{2}}$.

$4xy=b^2$. . . . . $y=\frac{1}{2}a\mp\frac{1}{2}(a^2-b^2)^{\frac{1}{2}}$.

(67.) $x^2+y^2=a^2$. . . . . $x=\frac{1}{2}b\pm\frac{1}{2}(2a^2-b^2)^{\frac{1}{2}}$.

$x+y=b$. . . . . . $y=\frac{1}{2}b\mp\frac{1}{2}(2a^2-b^2)^{\frac{1}{2}}$.

(68.) $x^2-xy=ay$. . . . . $x=\frac{ab+a\sqrt{ab}}{a-b}$.

$xy-y^2=bx$. . . . . $y=\frac{ab+b\sqrt{ab}}{a-b}$.

(69.) $x^2-y^2=a$. . . . . $x=\pm b\sqrt{\frac{a}{b^2-1}}$.

$\frac{x}{y}=b$. . . . . $y=\pm\sqrt{\frac{a}{b^2-1}}$.

Ans.

(70.) $x^3+y^3=133$, and $x+y=7$. $x=5$, or 2; $y=2$, or 5.

(71.) $x^3-y^3=117$, and $x-y=3$. $x=5$, or -2; $y=2$, or -5.

(72.) $x^4+y^4=1297$, and $xy=6$. $x=\pm 6$, &c.; $y=\pm 1$, &c.

(73.) $x^4-y^4=240$, and $x-y=2$. $x=4$, &c.; $y=2$, &c.

(74.) $x^2+y^2=a$, $x^2+z^2=b$, and $y^2+z^2=c$.

$$x=\pm\left(\frac{a+b-c}{2}\right)^{\frac{1}{2}};\ y=\pm\left(\frac{a-b+c}{2}\right)^{\frac{1}{2}};\ z=\left(\frac{-a+b+c}{2}\right)^{\frac{1}{2}}.$$

QUESTIONS PRODUCING QUADRATIC EQUATIONS.

(1.) Find a number which is less than its square by 6.
Ans. 3, or -2.

(2.) What is the number, to which if its square and cube be added, the sum is equal to thirteen times the number itself? Ans. 0, 3, or -4.

(3.) Find two square fields having the sum of their areas an acre, and the difference of their sides 8 perches.
Ans. The sides are 4 and 12 perches.

(4.) Find a number such that if it be increased by 5, and diminished by 2, the product of the results may be 60.
Ans. 7, or -10.

(5.) Find a number such that if it be diminished by 5, and increased by 2, the product of the results may be 60.
Ans. 10, or -7.

(6.) Required a number such that if 5 be taken from its double, and 11 from its treble, the product of the remainders may be 20. Ans. 5, or $1\frac{1}{6}$.

(7.) Required a number such that if $2a$ be added to it, and $2b$ subtracted from it, the product of the results may be $3a^2+2ab$. Ans. $a+2b$, or $-3a$.

(8.) Find a number such that if 12 be subtracted from its double, and the remainder multiplied by the number itself, the product shall be 14. Ans. 7, or -1.

(9.) Find a number such that if it be multiplied by n, and a subtracted from the product, the remainder, multiplied by n times the number itself, shall be equal to b^2.

Ans. $\frac{a\pm\sqrt{(a^2+4b^2)}}{2n}$.

(10.) Divide 20 into two such parts that the sum of their squares shall be 208. Ans. 12, and 8.

(11.) Divide 210 into two such parts that the one is the square of the other. Ans. 14, and 196.

(12.) Divide a into two parts such that the one is the square of the other. Ans. $\frac{-1 \pm \sqrt{(1+4a)}}{2}$.

(13.) Find two consecutive numbers whose product is 552. Ans. 23, and 24.

(14.) Divide 18 into two such parts that twice the square of the greater shall exceed three times the square of the less by 95. Ans. 11, and 7.

(15.) Divide 60 into two such parts that their product shall be to the sum of their squares in the ratio of 2 to 5. Ans. 20, and 40.

(16.) Divide 225 into two such parts that the sum of their square roots shall be 21. Ans. 144, and 81.

(17.) Find two numbers whose difference is $\frac{1}{12}$, and the difference of their reciprocals $\frac{1}{6}$.

Ans. $\frac{2}{3}$, and $\frac{3}{4}$; or $-\frac{2}{3}$, and $-\frac{3}{4}$.

(18.) Find two numbers whose sum is 52, and whose product is three times the cube of the less.

Ans. 4, and 48; or $-4\frac{1}{3}$, and $56\frac{1}{3}$.

(19.) Find two numbers, each consisting of two digits, and such that their difference shall be twice the unit figure of the greater, their sum five times the product of the digits of the greater, and their product 384. Ans. 16, and 24.

(20.) A grazier bought a number of sheep for £40, 16s., and sold them again at £1, 15s. 6d. per head; he gained 2s. more than one sheep cost him; how many did he buy?

Ans. 24.

(21.) A grazier bought a number of sheep for £60; reserving 15, he sold the remainder for £54, and gained 2s. per head; how many did he buy? Ans. 75.

(22.) Find four consecutive numbers such that if the first two be taken as the digits of a number, that number is the product of the other two. Ans. 1, 2, 3, 4; or 5, 6, 7, 8.

(23.) Find three consecutive numbers such that if the last be multiplied by the sum of the other two, the product is equal to the number expressed by the first and last taken as digits. Ans. 3, 4, and 5; or 0, 1, and 2.

(24.) A person bought a number of sheep for £150; and

if he had got 20 more for the same money they would have cost him 5s. a-piece less; required the number of sheep, and the price of each. Ans. 100 at 30s.

(25.) Divide a given line a, into two such parts that the rectangle contained by the whole and one of the parts may be equal to the square of the other.

Ans. $a\frac{3+\sqrt{5}}{2}$, and $a\frac{3-\sqrt{5}}{2}$.

(26.) Find a fraction which is greater than its square by $\frac{1}{4}$. Ans. $\frac{1}{2}$.

(27.) A person divided £5 equally among some poor people; but if he had given them 1s. a-piece less, he would have relieved 5 more; how many were there? Ans. 20.

(28.) A general having drawn up his army in the form of ten equal solid squares, finds he has 640 men to spare; but on diminishing the side of each square by 3 men, and attempting to form 12 such squares, he finds he wants 172 men to complete the last: of what number did his army consist?

Ans. 20,000.

(29.) Find two numbers whose sum is 7, and the sum of their fifth powers 1267. Ans. 3, and 4.

(30.) Find two numbers whose sum is 5, and the sum of their third powers 35. Ans. 2, and 3.

(31.) Find two numbers whose product is 8, and the difference of their third powers 56.

Ans. 2, and 4, or -2, and -4.

(32.) Find two numbers such that the cube of their sum shall exceed the sum of their cubes by 210, and the difference of their cubes shall exceed the cube of their difference by 90. Ans. 5, and 2.

(33.) Solve the last exercise in general terms, using a instead of 210, and b instead of 90.

Ans. $\sqrt[3]{\frac{(a+b)^2}{6(a-b)}}$, and $\sqrt[3]{\frac{(a-b)^2}{6(a+b)}}$

(34.) The product of two numbers is 80, and if one of them be increased by 6, and the other diminished by 3, the product of the results is the same.

Ans. 8, and 10, or -5, and -16.

(35.) Find two numbers whose sum is 8, and the sum of their fourth powers 706. Ans. 5, and 3.

(36.) Two trains start at the same time to perform a journey of 156 miles; one of them, by travelling a mile an hour faster than the other, arrives at its destination an hour before the other: at what rate did each travel?

Ans. 13 and 12 miles per hour.

(37.) A farmer bought three flocks of sheep for £197, 8s.; each sheep cost as many shillings as there were sheep in the other two flocks: how many were in each flock, there being 2 more in the second and 3 more in the third than in the first? Ans. 24, 26, and 27.

(38.) Two trains start at the same time from the opposite termini of a railway, and when they meet it is found that one has travelled 60 miles more than the other, and by continuing to travel at the same rates the one will finish the journey in 4 hours and the other in 9 hours from the time of meeting; what is the length of the railway, and the rate of travelling of each train?

Ans. Railway 300 miles, and the rates 20 and 30 miles per hour; or 12 miles, and the rates 6 and —4 miles per hour.

(39.) Two trains start at the same time from the opposite termini of a railway, and when they meet it is found that the one has travelled twice as many miles per hour as the number of hours they have travelled; that the other has travelled 8 miles an hour faster, and will finish the whole journey in 21 hours; find the length of the railway, and the rate of each train. Ans. Railway 672 miles, and the rates 24 and 32 miles per hour.

(40.) A general having drawn up his army in the form of 6 solid squares, finds he has 634 men to spare; but by making the squares hollow, leaving only the three outer ranks, he finds he can form 42 such squares, with 688 men to spare: find the number of his army. Ans. 40,000.

RELATIONS OF THE ROOTS OF QUADRATIC EQUATIONS.

60. Resuming the consideration of the general equation $ax^2+bx=c$, we may now proceed to point out some of the more important relations of the roots of quadratic equations. Dividing by a, we get $x^2+\frac{b}{a}x=\frac{c}{a}$, or putting $\frac{b}{a}=2m$, and $\frac{c}{a}=n$,

$x^2+2mx=n$, where m and n may be any numbers, whole or fractional, positive or negative.

61. As before, we get $x = -m \pm \sqrt{(m^2 + n)}$; from which we infer, that since the surd part may be either positive or negative, there must be two roots or values of the unknown quantity in every quadratic equation, but only two.

62. If n be positive, the surd part will be greater than m, and one root will therefore be positive and the other negative.

63. If n be negative, and less than m^2, the surd part will be less than m; and in this case the roots will be both positive when m is negative, but both negative when m is positive.

64. If n be negative, and equal to m^2, the surd part will vanish, and the roots will each be $-m$.

65. If n be negative, and greater than m^2, both roots are imaginary; but both roots are real when n is positive, or when it is negative, and not greater than m^2.

66. Since the two roots are $-m + \sqrt{(m^2 + n)}$, and $-m - \sqrt{(m^2 + n)}$, by adding these roots we get $-2m$; that is, the sum of the roots is always equal to the coefficient of the lower power with its sign changed.

67. Since by multiplying the one root by the other we get $-n$, the product of the two roots is always equal to the second member with its sign changed.

68. Since $x = -m + \sqrt{(m^2 + n)}$, and $x = -m - \sqrt{(m^2 + n)}$, or, by transposing,

$$x + m - \sqrt{(m^2 + n)} = 0, \text{ and } x + m + \sqrt{(m^2 + n)} = 0,$$

by multiplying, $x^2 + 2mx - n = 0$, or $x^2 + 2mx = n$, which is the original equation; and hence, if each root be attached to x with its sign changed, the product of the results put equal to 0 will be the original equation.

69. Conversely, if it be required to find the factors of any given quantity of the second degree, put it equal to 0, and find the roots of the equation; these roots being attached to x with their signs changed will be the factors required.

If the given quantity be of the form $ax^2 + bx + c$, the factors found as above would be those of $\frac{ax^2 + bx + c}{a}$, or of $x^2 + \frac{b}{a}x + \frac{c}{a}$; one of them must therefore be multiplied by a, or one of them by one factor of a, and the other by the other factor of a; or, in other words, they must be cleared of fractions.

EXERCISES.

Resolve the following expressions into factors:

(1.) $x^2 - 4x + 4$. . . . . Ans. $(x-3)(x-1)$.

(2.) $x^2 + x - 6$. . . . . $(x-2)(x+3)$.

(3.) $x^2 - 5x + 6$. . . . . $(x-3)(x-2)$.

(4.) $2x^2 - 3x - 5$. . . . . . . $(2x-5)(x+1)$.

(5.) $16x^2 - 18x - 9$. . . . . . $(2x-3)(8x+3)$.

(6.) $x + 2x^{\frac{1}{2}} - 15$. . . . . $(x^{\frac{1}{2}}+5)(x^{\frac{1}{2}}-3)$.

CHAPTER IX.

ARITHMETICAL PROGRESSION.

70. A series is said to be in *Arithmetical Progression*, or to be *equidifferent*, when each term after the first is found by adding a constant quantity to the term immediately preceding it; and this constant quantity is called the *Common Difference* of the series.

71. If the common difference be positive, the series is an ascending one; but if negative, it is a descending one. Thus 1, 3, 5, &c. is an ascending series, whose common difference is 2; and 12, 9, 6, &c. is a descending one, whose common difference is -3.

72. In every series, of whatever kind, the first and last terms are called the extremes, and the intermediate ones, the means.

73. Let us assume $t_1, t_2, t_3, t_4, \ldots t_{n-1}, t_n$; to express the terms of an arithmetical series, t_1, and t_n, being the extremes; and let n denote the number of terms, d the common difference, and s_n the sum of n terms.

Then since each term after the first is found by adding d to the preceding one, the series may be written thus:

$$t_1,\ t_1+d,\ t_1+2d,\ t_1+3d,\ \ldots\ t_1+(n-1)d.$$

Hence $\qquad t_n = t_1 + (n-1)d.$

This equation contains the four quantities t_n, t_1, n, and d, and by means of it any one of them may be found when the other three are known.

EXERCISES.

(1.) Prove that $t_1 = t_n - (n-1)d$.

(2.) Prove that $d = \frac{t_n - t_1}{n-1}$.

(3.) Prove that $n = \frac{t_n - t_1 + d}{d}$.

(4.) Find the 50th and the 150th terms of the series 1, 3, 5, &c. Ans. 99, and 299.

(5.) Find the 22d and 52d terms of the series 100, 97, 94, &c. Ans. 37, and -53.

(6.) Find the 17th and 54th terms of the series 80, 76, 72, &c. Ans. 16, and -132.

(7.) Find the 365th term of the series $x + 2x + 3x$, &c.
Ans. $365x$.

(8.) Find the 999th term of the series $a + x$, $a + 2x$, &c.
Ans. $a + 999x$.

(9.) Find the 47th term of the series $\frac{1}{3}$, 1, $1\frac{2}{3}$, &c.
Ans. 31.

(10.) The last term of an arithmetical series is 120, the number of terms 40, and the common difference 3; find the first term. Ans. 3.

(11.) The extremes of an arithmetical series are ·6 and 79, and the number of terms 50; find the common difference. Ans. 1·6.

(12.) The extremes of an arithmetical series are 2·5 and 121·3, and the common difference 1·2; find the number of terms. Ans. 100.

(13.) Find two arithmetical means between 4 and 13.
Ans. 7, and 10.

(14.) Find 20 arithmetical means between 1 and 2.
Ans. $1\frac{1}{21}$, $1\frac{2}{21}$, $1\frac{3}{21}$, &c.

(15.) Find an arithmetical series whose fourth term is 3, and the sum of the first and second also 3.
Ans. $1\frac{1}{5}$, $1\frac{4}{5}$, $2\frac{2}{5}$, 3, &c.

74. To find the sum of an arithmetical series, we have, according to the notation already adopted,

$$s_n = t_1 + (t_1 + d) + (t_1 + 2d) \;.\;.\;.\;.\;.$$
$$.\;.\;.\; + \{t_1 + (n-2)d\} + \{t_1 + (n-1)d\}.$$

Or $$s_n = \{t_1 + (n-1)d\} + \{t_1 + (n-2)d\} \;.\;.\;.$$
$$.\;.\;.\; (t_1 + 2d) + (t_1 + d) \quad + \quad t_1.$$

Hence by Addition,

$$2s_n = \{2t_1 + (n-1)d\} + \{2t_1 + (n-1)d\} \;.\;.\text{ to } n \text{ terms.}$$

Or $$2s_n = n\{2t_1 + (n-1)d\}.$$

$$\therefore \quad s_n = \frac{n}{2}\{2t_1 + (n-1)d\}.$$

Or $$s_n = \frac{n}{2}(t_1 + t_n).$$

It appears from the above that the sum of the extremes is equal to the sum of any two terms equally distant from the extremes.

EXERCISES.

(1.) Prove that $t_n = \frac{2s_n - nt_1}{n}$.

(2.) Prove that $t_1 = \frac{2s_n - nt_n}{n} = \frac{2s_n - n(n-1)d}{2n}$.

(3.) Prove that $n = \frac{2s_n}{t_1 + t_n} = \frac{d - 2t_1 \pm \sqrt{(d^2 - 4t_1d + 4t_1^2 + 8ds_n)}}{2d}$.

(4.) Find the sum of 45 terms of the series whose first term is 3, and the last 179. Ans. 4095.

(5.) Find the sum of 50 terms of the series 4, 7, 11, &c. Ans. 3875.

(6.) Find the sum of n terms, and thence the sum of 20 terms of the series 1, 3, 5, &c. Ans. n^2, and 400.

(7.) Find the sum of n terms, and thence the sum of 20 terms of the series 1, 2, 3, &c. Ans. $\frac{1}{2}n(n+1)$, and 210.

(8.) Find the sum of all the odd numbers between 100 and 1000. Ans. 247,500.

(9.) The sum of 20 terms of an arithmetical series is 1240, and the first term is 100; find the last term. Ans. 24.

(10.) The extremes of an arithmetical series are $\frac{1}{2}$ and

$12\frac{1}{2}$, and the sum 390; find the number of terms and the series. Ans. 60, and $\frac{59}{118}$, $\frac{83}{118}$, $\frac{107}{118}$, &c.

(11.) Given the common difference, double the first term, prove that the sum of n terms is $n^2 \times$ the first term; that is, when $d = 2\,t_1$, $s_n = n^2 t_1$. Thence find how far a falling body would descend in 11 seconds; it being known that it descends 16·1 feet in the first second, 48·3 in the next, 80·5 in the third, and so on. Ans. 1948·1 feet.

(12.) How far would a falling body descend in 17 seconds, and how far in the 17th second?

Ans. 4652·9, and 531·3 feet.

(13.) Given the sum of 3 terms $= 15$, and the sum of their squares $= 93$; to find the series.

Ans. $t_1 = 2$ or 8, and $d = \pm 3$.

(14.) A messenger sets out at the rate of 30 miles a-day, but falls off in his speed 4 miles daily. Four days afterwards, another sets off from the same place on the same route, travelling 50 miles the first day, but falling off like the first 4 miles daily. After what time will one overtake the other? Ans. $6\frac{2}{3}$ days.

GEOMETRICAL PROGRESSION.

75. A series is said to be in *Geometrical Progression*, or *Continual Proportion*, when each term after the first is found by multiplying the one immediately preceding it by a constant number, and this multiplier is called the *Common Ratio* of the series.

76. If the common ratio, without regard to its sign, be greater than unity, the series is an ascending one; but if it be less than unity, it is a descending one. Thus, 1, 3, 9, &c. is an ascending series whose common ratio is 3; and $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} +$ &c. is a descending one whose common ratio is $-\frac{1}{2}$.

77. Retaining the same notation as before, let us assume $t_1, t_2, t_3, \ldots t_{n-1}, t_n$, to express the terms of a geometrical series, t_1 and t_n being the extremes, and let n denote the number of terms, r the common ratio, s_n the sum of n terms, and s_∞ the sum of an infinite number of terms. The symbol ∞ being used to denote a quantity infinitely great.

Then, since each term after the first is found by multiplying the preceding one by r, the series may be written thus:

$$t_1, rt_1, r^2t_1, \quad . \quad . \quad . \quad . \quad . \quad r^{n-2}t_1, r^{n-1}t_1.$$

Hence $\quad t_n = r^{n-1}t_1.$

This equation contains the four quantities t_1, t_n, n, and r; and by means of it any one of them may be found when the other three are known. *The finding of n, however, requires the use of Logarithms.*

EXERCISES.

(1.) Prove that $t_1 = \frac{t_n}{r^{n-1}}$, and $r = \left(\frac{t_n}{t_1}\right)^{\frac{1}{n-1}}$.

(2.) Find the common ratio in the series 2, 6, 18, &c.
Ans. 3.

(3.) Find the common ratio in the series $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{9}$, &c.
Ans. $\frac{2}{3}$.

(4.) Find the 12th and 17th terms of the series $\frac{1}{2} - \frac{1}{4} + \frac{1}{8}$, &c.
Ans. $-\frac{1}{4096}$, and $\frac{1}{131072}$.

(5.) The 6th term of a series is $\frac{2}{243}$, and the common ratio $\frac{1}{3}$; find the first term.
Ans. 2.

(6.) The 1st term is 12, and the 10th $\frac{3}{128}$; find the series.
Ans. 12, 6, 3, &c.

(7.) Find the geometric mean between 9 and 16.
Ans. 12.

(8.) Insert two geometric means between 1 and $\frac{1}{8}$.
Ans. $\frac{1}{2}$ and $\frac{1}{4}$.

(9.) Insert two geometric means between 2 and -54.
Ans. -6 and 18.

(10.) Insert three geometric means between 2 and 32.
Ans. 4, 8, 16; and -4, 8, -16.

(11.) Insert four geometric means between 4 and $\frac{1}{8}$.
Ans. 2, 1, $\frac{1}{2}$, $\frac{1}{4}$.

(12.) The sum of the first two terms is $4\frac{1}{2}$, and the sum of the next two $1\frac{1}{8}$; find the series.
Ans. $t_1 = 3$ or 9, and $r = \pm\frac{1}{2}$.

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78. To find the sum of a geometric series, we have

$$s_n = t_1 + rt_1 + r^2t_1 \ldots + r^{n-2}t_1 + r^{n-1}t_1.$$

Multiplying by r, $\quad rs_n = rt_1 + r^2t_1 + r^3t_1 \ldots + r^{n-1}t_1 + r^nt_1.$

Subtracting, $\quad rs_n - s_n = r^nt_1 - t_1.$

Hence $\quad s_n = t_1\frac{r^n - 1}{r - 1} = t_1\frac{1 - r^n}{1 - r}.$

The first form is best adapted for ascending series; the second is derived from it by multiplying numerator and denominator by — 1, and is best adapted for descending series.

79. If r be less than unity, the quantity r^n becomes less and less as n increases; and by taking n large enough, it may be made less than any quantity that can be assigned. Hence, if n be taken infinite, r^n becomes $= 0$, and therefore the formula

$$s_n = t_1 \frac{1 - r^n}{1 - r} \text{ becomes}$$

$$s_\infty = \frac{t_1}{1 - r}.$$

From which it appears that to find the sum of an infinite number of terms of a descending geometrical series, we have only to divide the first term by the excess of unity above the common ratio. However great a number of terms of the series be taken, their sum will never actually amount to $\frac{t_1}{1-r}$, but may be made to approach as near to it as we please.

EXERCISES.

Find the sum of each of the following series:

(1.) 1, 2, 4, 8, &c., to 8 terms. . . . Ans. 255.

(2.) 4, 3, $\frac{9}{4}$, &c., to 10 terms. . . . . . . $15\frac{6487}{65536}$.

(3.) $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c., to infinity. . . . . . . 1.

(4.) 1, $-\frac{1}{2}$, $\frac{1}{4}$, &c., to 9 terms and to infinity. . $\frac{171}{256}$ & $\frac{2}{3}$.

(5.) ·333, &c., to infinity. . . . . $\frac{1}{3}$.

(6.) ·4545, &c., to infinity. . . . . . . $\frac{5}{11}$.

(7.) ·234234, &c., to infinity. . . . . . . $\frac{26}{111}$.

(8.) $\frac{\sqrt{2}+1}{\sqrt{2}-1}$, $\frac{1}{2-\sqrt{2}}$, $\frac{1}{2}$, &c., to infinity. . . . $4 + 3\sqrt{2}$.

(9.) The first term of an infinite geometric series is 1, and each term is equal to the sum of all that follow it; find the series. Ans. 1, $\frac{1}{2}$, $\frac{1}{4}$, &c.

(10.) The sum of an infinite geometric series is $\frac{8}{21}$, and the sum of the first two terms is $\frac{1}{6}$; find the series.

Ans. $\frac{2}{3}$, $-\frac{1}{2}$, $\frac{3}{8}$, &c.; or $\frac{2}{21}$, $\frac{6}{84}$, $\frac{18}{336}$, &c.

(11.) There are 4 numbers in arithmetical progression, which, being increased respectively by 1, 1, 2, and $4\frac{1}{2}$, are in geometrical progression; find the numbers.

Ans. 3, 5, 7, 9.

(12.) Find the nth term, and the sum of n terms of the series $a+a$, $2a+a^2$, $3a+a^3$, $4a+a^4$, &c.

Ans. $na+a^n$, and $\frac{1}{2}n(n+1)a+a\frac{a^n-1}{a-1}$.

Find the nth term and the sum of n terms of each of the following series:

(13.) 1, 3, 7, 15, &c. . Ans. 2^n-1, & $2(2^n-1)-n$.

(14.) -1, 1, 5, 13, &c. . . 2^n-3, & $2(2^n-1)-3n$.

(15.) 4, 6, 10, 18, &c. . . . 2^n+2, & $2(2^n-1)+2n$.

(16.) 2, 6, 14, 30, &c. . . . $2^{n+1}-2$, & $4(2^n-1)-2n$.

(17.) 1, 2, 5, 12, &c. . . 2^n-n, & $2(2^n-1)-\frac{1}{2}n(n+1)$.

(18.) $\frac{2}{3}$, $\frac{5}{9}$, $\frac{11}{27}$, $\frac{29}{81}$, &c. . . . $\frac{3^{n-1}+2}{3^n}$, & $\frac{n}{3}+1-\frac{1}{3^n}$.

(19.) A horse is sold for a farthing for the first nail in his shoes, 2 farthings for the next, 4 for the next, and so on, doubling for every nail; find the price of the horse.

Ans. £4473924, 5s. $3\frac{3}{4}$d.

(20.) Suppose it were possible to pay 1 grain of wheat for the first square of a chess-board, 2 for the next, 4 for the next and so on, doubling for each of the 64 squares; how many bushels would it amount to, allowing 7680 grains to the pint? And supposing Europe to contain 1,000,000 square miles of arable land, how many years would it take to grow it if the whole were in crop, and each acre produced on an average 40 bushels per annum?

Ans. 37529996894754 bush. and 1466 years.

CHAPTER X.

PERMUTATIONS AND COMBINATIONS.

80. *Permutations* denote the different *orders* in which any quantities may be arranged; *Combinations* denote the different collections that may be formed out of them, without regard to the *order* in which they are placed. Thus ab and ba form different permutations, but only one combination.

81. If there be n things, a, b, c, d, &c., then, by taking them one by one, the number of permutations formed will be n. By placing a before each of the other $(n-1)$ things, there will be $(n-1)$ permutations with a standing first; also in like manner $(n-1)$ with b standing first; and so on for the whole n quantities. Hence there are

$n(n-1)$ permutations taken two together.

Suppose a to be removed, there are $(n-1)$ things remaining, and of these there are

$(n-1)(n-2)$ permutations taken two together.

By prefixing a to each of these, there will be $(n-1)(n-2)$ permutations of three with a standing first; and similarly $(n-1)(n-2)$ with b standing first; and so on for each of the n quantities. There are therefore

$n(n-1)(n-2)$ permutations taken three together.

Collecting these results, and taking P_1, P_2, P_3, . . . P_r, to denote the numbers of permutations of n things taken 1, 2, 3, . . . r, together respectively, we get

$$P_1 = n$$
$$P_2 = n(n-1)$$
$$P_3 = n(n-1)(n-2).$$

Similarly, it might be shown that

$$P_4 = n(n-1)(n-2)(n-3)$$
$$P_5 = n(n-1)(n-2)(n-3)(n-4);$$

and hence we may infer, that generally

$$P_r = n(n-1)(n-2)(n-3) \ldots (n-r+1).$$

82. Assuming that this law holds true when $(r-1)$ things are taken together, then

$$P_{r-1} = n(n-1)(n-2) \ \ldots \ (n-r+2);$$

but if one of the things, viz. a, be omitted, it must be equally true that the number of permutations of the remaining $(n-1)$ things taken $(r-1)$ together, is

$$(n-1)(n-2)(n-3) \ \ldots \ (n-r+1)$$

found by putting $(n-1)$ for n in the preceding equation.

Now, by prefixing a to each of these last permutations, their number would remain the same, but there would be r taken together instead of $r-1$.

In like manner it may be shown that the same number of permutations may be formed of these n things taken r together, in which b stands first, and so on for each of the others. If therefore

$$P_{r-1} = n(n-1)(n-2) \ \ldots \ (n-r+2),$$

it follows that

$$P_r = n(n-1)(n-2)(n-3) \ \ldots \ (n-r+1).$$

That is, if the assumed law be true for any one value of r, it must be true for the next higher value. But it has been proved to hold true when $r=2$, and $r=3$; therefore it is true when $r=4$, and if true for 4, then also for 5; and so on for any number.

If $r=n$, the last factor in the above formula becomes 1, therefore denoting the number of permutations of n things taken altogether by P_n, we have

$$P_n = n(n-1)(n-2) \ \ldots 3 \,.\, 2 \,.\, 1,$$
$$\text{or} \quad = 1 \,.\, 2 \,.\, 3 \,.\, 4 \ldots \ldots \ n.$$

EXERCISES.

(1.) Find the number of permutations of 8 things taken 2 together, 3 together, 4 together, and all together.

Ans. 56, 336, 1680, and 40320.

(2.) In how many different ways may a party of 12 be arranged at table? Ans. 479001600.

(3.) The number of permutations of n things, taken 4 together $= 8$ times the number taken 3 together; find n.
Ans. 11.

(4.) In how many ways may a class of 9 boys be arranged? Ans. 362880.

(5.) In the permutations formed out of the eight letters a, b, c, d, e, f, g, h, taken altogether, how many begin with ab? how many with abc? and how many with $abcd$?
Ans. 720, 120, and 24.

(6.) The number of permutations of 18 things taken r together $= 12$ times the number taken $r - 1$ together; find r. Ans. $r = 7$.

(7.) In how many ways can 8 persons be seated at a round table, so that all shall not have the same neighbours twice? Ans. 2520.

(8.) Find the number of changes which can be rung with a peal of 7 bells, and what difference will be made in the number by the absence of one ringer? Ans. 5040, & 4320.

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83. Again, let $C_1, C_2, C_3 \ldots C_r$, denote the number of combinations of n things taken 1 by 1, 2 by 2, 3 by 3 . . . r by r together;

then it is evident that $C_1 = n$.

And since each combination of 2 things, as ab, admits of 2 permutations, ab, and ba, it follows that

$$C_2 = \frac{P_2}{2}.$$

$$\text{But (§ 81.) } P_2 = n(n-1).$$

$$\therefore \quad C_2 = \frac{n(n-1)}{1 \cdot 2}.$$

Similarly, since each combination of 3 things admits of $3 \cdot 2 \cdot 1$ permutations,

$$C_3 = \frac{P_3}{1 \cdot 2 \cdot 3}.$$

$$\text{But (§ 81.) } P_3 = n(n-1)(n-2).$$

$$\therefore \quad C_3 = \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}.$$

And, generally, since each combination of r things admits of $1 \cdot 2 \cdot 3 \ldots r$ permutations,

$$C_r = \frac{P_r}{1 . 2 . 3 \ldots r}.$$

But (§ 81.) $P_r = n(n-1)(n-2) \ldots (n-r+1)$.

$$\therefore \quad C_r = \frac{n(n-1)(n-2) \ldots \ldots (n-r+1)}{1 . 2 . 3 \ldots \ldots r}.$$

84. On examining this last formula, it will be observed that the number of factors is the same in the numerator and denominator of the second member, and in both is always $= r$, the number of things taken together.

85. Also, since of n things if r be taken, there will be $n - r$ remaining; for every combination of r things, there will always be another of $n - r$ things remaining: hence

$$C_r = C_{n-r}.$$

EXERCISES.

(1.) Find the number of combinations of 12 things taken 2 together, 3 together, and 6 together.

Ans. 66, 220, and 924.

(2.) Find the number of combinations of 20 things taken 18 together. Ans. 190.

(3.) Find the number of combinations of 50 things taken 4 together, 10 together, 40 together, and 46 together.

Ans. 230300, and 10272278170.

(4.) How many different hands may a person hold at the game of whist? Ans. 635013559600.

(5.) How many combinations may be formed of the letters in the word "Equations" taken 5 together? and in how many of them will the first letter E occur?

Ans. 126, and 70.

(6.) From a company of 30 soldiers, 5 are selected every day to act as sentinels; on how many days may a different selection be made? and on how many of these will any particular man be on duty? Ans. 142506 days, and 23751.

(7.) An innkeeper engaged to provide a dinner every day for 6 persons, to be selected out of a party of 12, so long as the same 6 should not meet twice; what would be the amount of his bill at half-a-crown each per day?

Ans. £693.

(8.) How many different sums may be formed with a sovereign, a half-sovereign, a crown, a half-crown, a shilling, sixpence, and a penny? Ans. 127.

(9.) Find how many triangles may be formed by joining the angular points of a polygon of n sides.

Ans. $\frac{1}{6}n(n-1)(n-2)$.

(10.) Expand 2^n in the form $(1+1)^n$, and thence prove that the total number of combinations of n things taken $1, 2, 3 \ldots n$ together $= 2^n - 1$.

(11.) The total number of combinations of $2n$ things $=$ 33 times the total number of combinations of n things; find n. Ans. $n = 5$.

(12.) Out of a committee of 12, how many sub-committees may be formed, each consisting of not more than 8?

Ans. 3796.

(13.) Let P denote the number of permutations of n things taken all together, when some of them recur, and prove that if a recur r times; b, s times; c, t times, &c.

$$P = \frac{1 \cdot 2 \cdot 3 \ldots n}{1 \cdot 2 \cdot 3 \ldots r,\ 1 \cdot 2 \cdot 3 \ldots s,\ 1 \cdot 2 \cdot 3 \ldots t, \text{\&c.}}$$

(14.) Find the number of permutations that can be formed from the letters of each of the following words taken all together:—Algebra, Permutations, Combinations, Susannah, Philippopolis, and Mississippi. Ans. 2520; 239500800; 119750400; 5040; 10810800; and 34650.

CHAPTER XI.

INDETERMINATE COEFFICIENTS.

86. The general principle of the method of Indeterminate Coefficients consists in assuming certain letters to express the unknown coefficients in an expression for some required quantity, the general form of which expression is known; then, according to the nature of the investigation, two expressions of the same form are found, which are rendered *identical* by making the coefficients of the corresponding terms equal; and by means of the series of equations thus obtained the values of the assumed coefficients are determined.

87. That an equation cannot be *identical*, that is, true for all values whatever of the unknown quantity, unless the coefficients of the like powers are equal, may be thus shown:

Let $A+Bx+Cx^2+$, &c. $=a+bx+cx^2+$, &c., be an identical equation; then, by transposition,

$$(B-b)x+(C-c)x^2+, \text{ \&c.} = a-A.$$

And since both a and A are constant quantities, $a-A$ must be a constant quantity also. But unless the coefficients of x, x^2, &c. be each $=0$, the left-hand member would have different values, according to the value assigned to x, and consequently the right-hand member would have different values also, which is impossible. Therefore, $B-b=0$, or $B=b$. $C-c=0$, or $C=c$, &c. &c.

Example 1. Let it be required to divide $1+x$ by $1-2x$ by this method. A little consideration will show that the quotient will consist of ascending powers of x, with certain coefficients unknown; or this may be ascertained by finding a few terms of the quotient by actual division. Assuming, therefore,

$$\frac{1+x}{1-2x} = A+Bx+Cx^2+Dx^5+, \text{ \&c.; we get}$$

by multiplication,
$$1+x = \begin{array}{c|c|c|c} A+B & x+C & x^2+D & x^3+, \text{ \&c.} \\ -2A & -2B & -2C & -, \text{ \&c.,} \end{array}$$

an equation which must be identical; and therefore,

$A=1$, $B-2A=1$, $C-2B=0$, $D-2C=0$, &c.;

therefore, $B=3$, $C=6$, $D=12$, &c.

Hence, by substitution,

$$\frac{1+x}{1-2x} = 1+3x+6x^2+12x^0+\ldots 3\cdot2^{n-2}x^{n-1}.$$

Example 2. Let it be required to find the sum of n terms of the series $1^2+2^2+3^2+4^2+$, &c., to n^2.

Since the sum is evidently less than

$$n^2+n^2+n^2+, \text{ \&c., to } n \text{ terms, or than } n^3,$$

in expressing this sum in terms of n with unknown coefficients, we need not use any higher power than the third.

We shall have therefore,

$$1^2 + 2^2 + \ldots n^2 = An^3 + Bn^2 + Cn.$$

Changing n into $(n + 1)$,

$$1^2+2^2+\ldots n^2+(n+1)^2=A(n+1)^3+B(n+1)^2+C(n+1),$$

and by subtraction,

$$(n + 1)^2 = A(3n^2 + 3n + 1) + B(2n + 1) + C.$$

Hence, by equalizing the coefficients of like powers of n,

$$\text{we get}\quad A = \tfrac{1}{3};\ B = \tfrac{1}{2};\ \text{and } C = \tfrac{1}{6};$$

$$\therefore \text{ by substitution, } 1^2+2^2 \ldots n^2 = \tfrac{1}{3}n^3+\tfrac{1}{2}n^2+\tfrac{1}{6}n,$$
$$= \tfrac{1}{6}n(n + 1)(2n + 1).$$

NOTE. If we assume powers of n higher than the third, the co-efficients of those powers will be found each $= 0$.

Example 3. Lastly, let it be required to extract the square root of $1 + x$.

Assume $(1 + x)^{\frac{1}{2}} = A + Bx + Cx^2 + Dx^3 +$, &c.;
then, by squaring,

$$\begin{array}{rl|l|l|l} 1 + x = & A^2 + AB & x + AC & x^2 + AD & x^3 +, \&c. \\ & + AB & + B^2 & + BC & +, \&c. \\ & & + AC & + BC & +, \&c. \\ & & & + AD & +, \&c. \end{array}$$

And by putting the coefficients of like powers equal as before, we get

$$A = \pm 1,\ B = \pm \tfrac{1}{2},\ C = \mp \frac{1}{2.4},\ D = \pm \frac{1.3}{2.4.6},\ \&c.$$

$$\text{Hence } (1+x)^{\frac{1}{2}} = \pm 1 \pm \tfrac{1}{2}x \mp \frac{1}{2.4}x^2 \pm \frac{1.3}{2.4.6}x^3 \mp, \&c.$$

EXERCISES.

Find by this method the series equivalent to each of the following fractions:

(1.) $\frac{1}{1-x}$. . . . . . Ans. $1+x+x^2+x^3+$, &c.

(2.) $\frac{1}{1+x}$. . . . . $1-x+x^2-x^3+$, &c.

(3.) $\frac{a}{a-x}$. . . . . . Ans. $1+\frac{x}{a}+\frac{x^2}{a^2}+\frac{x^3}{a^3}+$, &c.

(4.) $\frac{1+x+x^2}{1+2x+x^2}$. . . . . . $1-x+2x^2-3x^3+$, &c.

Sum each of the following series to n terms:

(5.) $1+2+3+$, &c. . . Ans. $\frac{1}{2}n(n+1)$.

(6.) $1^2+3^2+5^2+$, &c. . . $\frac{1}{6}(2n-1)2n(2n+1)$.

(7.) $1^3+3^3+5^3+$, &c. . . $2n^4-n^2=n^2(2n^2-1)$.

(8.) $1.2+2.3+3.4+$, &c. . . $\frac{1}{3}n(n+1)(n+2)$.

(9.) $1.2.3+2.3.4+3.4.5+$, &c. $\frac{1}{4}n(n+1)(n+2)(n+3)$.

(10.) $2^3+4^3+6^3+$, &c. . . $2n^2(n+1)^2$.

(11.) $1^2.2^2+2^2.3^2+3^2.4^2+$, &c. $\frac{1}{5}n^5+n^4+\frac{5}{3}n^3+n^2+\frac{2}{15}n$.

(12.) Prove that $(1+2+3+\ldots\ldots n)^2=(1^2-2^2+3^2-\ldots\pm n^2)^2=1^3+2^3+3^3+\ldots n^3$.

Resolve each of the following quantities into its component factors of the first degree:

(13.) $6x^2+7x-20$. . . . Ans. $(3x-4)(2x+5)$.

(14.) $3x^2-5x-50$. . . . . . $(x-5)(3x+10)$.

(15.) x^2-4x+4. . . . . . $(x-1)(x-3)$.

(16.) $6x^3-5x^2+x$. . . . . . $x(2x-1)(3x-1)$.

Decompose each of the following fractions into its component partial fractions:

(17.) $\frac{x^2+16}{x^2-16}$. . . . . . Ans. $\frac{x}{x+4}$, $\frac{4}{x-4}$.

(18.) $\frac{2x^2+7}{x^2-x-6}$. . . . . . . $\frac{x-1}{x+2}$, $\frac{x+2}{x-3}$.

(19.) $\frac{4x^2}{(2x-1)^3}$. . . . . . . $\frac{2x}{(2x-1)^3}$, $\frac{1}{(2x-1)^2}$, $\frac{1}{2x-1}$.

(20.) $\frac{2x^2-6}{x^3+2x^2-3x}$. . . . . . $\frac{2}{x}$, $\frac{-1}{x-1}$, $\frac{1}{x+3}$.

(21.) $\frac{2x^2+2y^2}{x^2-y^2}$. . . . . . . $\frac{x+y}{x-y}$, and $\frac{x-y}{x+y}$.

Ans.

(22.) $\dfrac{18x^2 - 22x + 6}{(x-1)(2x-1)(3x-1)}$. $\dfrac{1}{x-1}$, $\dfrac{2}{2x-1}$, & $\dfrac{3}{3x-1}$.

(23.) $\dfrac{x^3 - x^2 + 2x + 2}{(x-1)^4}$. . $\dfrac{1}{x-1}$, $\dfrac{2}{(x-1)^2}$, $\dfrac{3}{(x-1)^3}$, & $\dfrac{4}{(x-1)^4}$.

Expand each of the following quantities into a series:

(24.) $(a-x)^{\frac{1}{2}}$. Ans. $\pm a^{\frac{1}{2}}(1-\frac{1}{2}.\frac{x}{a}-\frac{1}{2.4}.\frac{x^2}{a^2}-\frac{1.3}{2.4.6}.\frac{x^3}{a^3}-$, &c.)

(25.) $(1+x^2)^{\frac{1}{2}}$. . $1+\frac{1}{2}x^2-\frac{1}{2.4}x^4+$ - - - -

(26.) $(a+x)^{\frac{1}{3}}$. . . $a^{\frac{1}{3}}(1+\frac{x}{3a}-\frac{2x^2}{3.6.a^2}+\frac{2.5x^3}{3.6.9a^3}-$, &c.)

CHAPTER XII.

SUMMATION OF SERIES, AND PILING OF BALLS.

88. When an infinite series is such, that if the sum of a large number of terms be taken as the sum of the whole, the error admits of being made less than any quantity that can be assigned, however small, that series is said to be *convergent;* all others are *divergent.* Thus the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+$, &c. is convergent,

but $1+2+4+8+$, &c. is divergent.

Ex. 1. Let it be required to find the sum of the infinite series,

$$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\frac{1}{4.5.6}+, \text{\&c.}$$

Assume $\quad A=\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+$, &c. . . (1.)

then, by transposition, $A-\frac{1}{2}=\frac{1}{2.3}+\frac{1}{3.4}+$, &c. (2.)

by subtracting (2.) from (1.), $\frac{1}{2}=\frac{2}{1.2.3}+\frac{2}{2.3.4}+$, &c.

and dividing by 2, $\qquad \frac{1}{4}=\frac{1}{1.2.3}+\frac{1}{2.3.4}+$, &c.

Hence, to sum a series by this method: assume A = a series formed by omitting the last factor from the denominator of each term of the given series, and proceed as above.

To find the sum of n terms of the same series:

Assume, as before,

$$A = \frac{1}{1.2} + \frac{1}{2.3} + \dots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)},$$

transposing,

$$A - \tfrac{1}{2} = \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{(n+1)(n+2)};$$

and subtracting,

$$\tfrac{1}{2} = \frac{2}{1.2.3} + \frac{2}{2.3.4} + \dots + \frac{2}{n(n+1)(n+2)} + \frac{1}{(n+1)(n+2)};$$

transposing, and dividing by 2,

$$\tfrac{1}{4} - \frac{1}{2(n+1)(n+2)} = \frac{1}{1.2.3} + \frac{1}{2.3.4} + \dots + \frac{1}{n(n+1)(n+2)}.$$

It will be observed that in this case the series to which A is assumed equal requires to be carried out one term beyond the nth, and the rest of the work is nearly the same as before; if n be taken infinite in the result, the sum $= \frac{1}{4}$.

Ex. 2. Let it be required to sum the infinite series,

$$\frac{1}{2.4.6} + \frac{3}{4.6.8} + \frac{5}{6.8.10} +, \text{\&c.}$$

Assume $\quad A = \dfrac{1}{2.4} + \dfrac{3}{4.6} + \dfrac{5}{6.8} +, \text{\&c.} \dots (1.)$

transposing, $A - \frac{1}{8} = \dfrac{3}{4.6} + \dfrac{5}{6.8} + \dfrac{7}{8.10} +, \text{\&c.}$

or, $\quad A - \frac{1}{8} = \dfrac{1}{4.6} + \dfrac{3}{6.8} + \dfrac{5}{8.10} +, \text{\&c.}$

$$+ \frac{2}{4.6} + \frac{2}{6.8} + \frac{2}{8.10} +, \text{\&c.}$$

But the sum of this last series found as in Ex. 1. $= \frac{1}{4}$,

$\therefore$ transposing, $A - \frac{3}{8} = \dfrac{1}{4.6} + \dfrac{3}{6.8} + \dfrac{5}{8.10} +, \text{\&c.} \dots (2.)$

subtracting (2) from (1), $\frac{5}{8} = \frac{4}{2.4.6} + \frac{12}{4.6.8} + \frac{20}{6.8.10} +$, &c.

and dividing by 4, $\frac{5}{32} = \frac{1}{2.4.6} + \frac{3}{4.6.8} + \frac{5}{6.8.10} +$, &c.

In a similar manner the sum of n terms may be found.

Find the sum of n terms, and of an infinite number of terms of each of the following series:

(1.) $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} +$, &c. Ans. $s_n = \frac{n}{n+1}$; $s_\infty = 1$.

(2.) $\frac{1}{1.2.3} + \frac{1}{2.3.4} +$, &c. . $s_n = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$; $s_\infty = \frac{1}{4}$.

(3.) $\frac{1.2.3}{1.2.3} + \frac{1.2.3}{2.3.4} +$, &c. . $s_n = \frac{3}{2} - \frac{3}{(n+1)(n+2)}$; $s_\infty = \frac{3}{2}$.

(4.) $\frac{1}{1.2.3.4} + \frac{1}{2.3.4.5} +$, &c.

$$s_n = \frac{1}{18} - \frac{1}{3(n+1)(n+2)(n+3)}; \quad s_\infty = \frac{1}{18}.$$

(5.) $\frac{1}{2.6} + \frac{1}{4.8} + \frac{1}{6.10} +$, &c.

$$s_n = \frac{5}{16} - \frac{2n+3}{8(n+1)(n+2)}; \quad s_\infty = \frac{5}{16}.$$

(6.) $\frac{1}{3.6} - \frac{1}{6.8} + \frac{1}{9.10} - \frac{1}{12.12} +$, &c.

$$s_n = \frac{1}{24} \pm \frac{1}{12(n+1)(n+2)}; \quad s_\infty = \frac{1}{24}.$$

(7.) $\frac{1}{1.3} - \frac{2}{3.5} + \frac{3}{5.7} -$, &c.

$$s_n = \frac{1}{4} \pm \frac{1}{4(2n+1)}; \quad s_\infty = \frac{1}{4}.$$

(8.) Since $1^3 = 1$, $2^3 = 3+5$, $3^3 = 7+9+11$, $4^3 = 13 + 15 + 17 + 19$, &c., write down n^3 according to the same law, and verify the result.

Ans. $n^3 = (n^2 - n + 1) + (n^2 - n + 3) \ldots + (n^2 + n - 1) = n^3$.

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89. When cannon balls are piled in the form of a pyramid, the base may be either a *square* or an *equilateral triangle*. In the former case, if n denote the number of balls

in one side of the base, the numbers in the successive layers will be

$$n^2,\ (n-1)^2,\ (n-2)^2,\ \&c.$$

terminating in a single ball at the top.

$$\therefore \text{ total number of balls } = 1^2+2^2+3^2+\ldots n^2,$$
$$\text{or (§ 87. Exam. 2.)} \quad = \tfrac{1}{6}n\,(n+1)\,(2n+1).$$

90. If the base be an equilateral triangle, and if n denote the number of balls in one of its sides, the number in the bottom layer will be

$$1+2+3+\ldots+n = \tfrac{1}{2}n(n+1)\ldots \text{(Exer. 5., p. 80.)}$$

By taking n successively equal to 1, 2, 3, &c., in the expression $\frac{1}{2}n\,(n+1)$, we find the number of balls in the several layers, commencing at the top, to be

$$\frac{1.2}{2},\ \frac{2.3}{2},\ \frac{3.4}{2},\ \&c.$$

$$\therefore \text{ total number of balls} = \tfrac{1}{2}\{1.2+2.3+3.4+\ldots n(n+)\},$$
$$\text{or (Exer. 8. p. 80.)} \quad = \tfrac{1}{6}n(n+1)\,(n+2).$$

91. In addition to these two kinds of piles, another is sometimes used, in which the base is a rectangle with unequal sides. In this case the pile will terminate in a single row, containing one ball more than the difference of the numbers in the greater and less sides of the base. Hence, if d denote this difference, the numbers in the several layers will be

$$d+1,\ 2(d+2),\ 3(d+3),\ \&c., \text{ and in general } n(d+n).$$

$$\begin{aligned}\therefore \text{ total number of balls} &= (d+1)+2(d+2)+\ldots n(d+n)\\ &= 1^2+2^2+3^2+\ldots n^2\\ &\quad + d(1+2+3+\ldots n).\\ \text{(§ 87. Exam. 2.)} \quad &= \tfrac{1}{6}n(n+1)\,(2n+1)\\ \text{and (Ex. 7. p. 68.)} \quad &\quad + \tfrac{1}{2}dn(n+1)\\ &= \tfrac{1}{6}n(n+1)\,(2n+1+3d).\end{aligned}$$

EXERCISES.

(1.) Find the number of balls in a square pile, the number in one side of the base being 60. Ans. 73810.

(2.) Find the number in a triangular pile, with 60 in one side of the base. Ans. 37820.

(3.) Find the number in a rectangular pile, the numbers in the greater and less sides of the base being 70 and 50 respectively. Ans. 68425.

(4.) Find the numbers in two incomplete piles, the one square and the other triangular; each having 20 layers, and 20 balls in the side of the uppermost layer.

Ans. 18070, and 9330.

(5.) Find the number in an incomplete rectangular pile of 20 layers, having 20 balls in the less side, and 30 in the greater of the uppermost layer. Ans. 23970.

CHAPTER XIII.

MISCELLANEOUS EXERCISES.

(1.) If A and B can perform a piece of work in c days; A and C in b days; and B and C in a days; in how many days will each alone perform it? and how long will they take all working together?

Ans. $\frac{2abc}{ab+ac-bc}$; $\frac{2abc}{ab-ac+bc}$; $\frac{2abc}{-ab+ac+bc}$; & $\frac{2abc}{ab+ac+bc}$.

(2.) Find the time between 4 and 5 o'clock, when the hour and minute hands of a watch are exactly together.

Ans. $21\frac{9}{11}$ minutes past 4.

(3.) Find the time after h o'clock, when the hour and minute hands are distant m of the minute divisions from each other. Ans. $\frac{12}{11}(5h \pm m)$.

(4.) Divide a into three parts, such that if the first be divided by m, the second by n, and the third by p, the quotients shall be all equal.

Ans. $\frac{ma}{m+n+p}$; $\frac{na}{m+n+p}$; and $\frac{pa}{m+n+p}$.

(5.) A and B set out at the same time from two places 333 miles distant, intending to meet on the road; A travels at the rate of 7 miles in 2 hours, and B 8 miles in 3 hours: in what time will they meet, and how far will each then have travelled?

Ans. 54 hours; A 189 miles, and B 144 miles.

(6.) A allows B a start of 2 hours, and then sets out in pursuit of him. B travels uniformly at a certain rate; but after A has travelled 3 hours, he discovers that B is travelling $2\frac{1}{2}$ miles per hour faster than himself; A therefore now doubles his speed, and overtakes B in 4 hours more: find their rates of travelling, and the distance travelled.

Ans. A's rate at first $11\frac{1}{4}$ miles per hour; B's, $13\frac{3}{4}$; and the distance $123\frac{3}{4}$ miles.

(7.) A hare is 50 of its own leaps before a greyhound, and takes 3 leaps for every 2 of the greyhound; but the latter passes over as much ground in 1 leap as the former in 2: how many leaps will each take before the hare is caught?

Ans. The greyhound 100, and the hare 150.

(8.) Bought $m + n$ sheep at s shillings per head; sold m of them at a profit of 10 per cent., and the rest n at a profit of 20 per cent.: how much is gained on the whole, and how much per cent.?

Ans. $\frac{s}{10}(m+2n)$ shillings; and $10\frac{m+2n}{m+n}$ per cent.

(9.) Find a number consisting of two places of figures which is equal to 4 times the sum of its digits; but if 6 be added to it, it will then be equal to 7 times the unit figure.

Ans. 36.

(10.) A person has two casks, each containing a certain quantity of liquid; he pours from the first into the second as much as the second contained at first; then from the second into the first as much as was left in the first; and, lastly, from the first into the second as much as was left in the second; there are now exactly 24 gallons in each: how much did each contain at first?

Ans. 33 gallons, and 15.

(11.) Find two numbers in the ratio of m to n, whose difference is equal to their product. Ans. $\frac{m-n}{n}$, and $\frac{m-n}{m}$.

(12.) The difference of two numbers is $\frac{1}{m}$ of their sum, and $\frac{1}{n}$ of their product; what are the numbers?

Ans. $\frac{2n}{m-1}$, and $\frac{2n}{m+1}$.

(13.) From a cask of wine containing a gallons, b gallons are drawn off, and the cask filled up with water; find how much wine remains in the cask after this has been repeated n times. Ans. $\frac{(a-b)^n}{a^{n-1}}$.

(14.) The number of permutations of n things taken r together $=5$ times the number taken $(r-1)$ together; and the number of combinations taken r together $=$ the number taken $(r-1)$ together; find n and r.

Ans. $n=9$; and $r=5$.

(15.) Expand $(a^3-x^3)^{\frac{1}{3}}$ into a series, both by the binomial theorem and by the method of indeterminate coefficients. Ans. $a(1-\frac{x^3}{3a^3}-\frac{2x^6}{3.6a^6}-\frac{2.5x^9}{3.6.9a^9}-$, &c.)

(16.) Find the sum of n terms, and of an infinite number of terms of the series:—

$$\frac{4}{1.2.3}-\frac{6}{2.3.4}+\frac{8}{3.4.5}-, \&c.$$

Ans. $s_n=\frac{1}{2}\pm\frac{1}{(n+1)(n+2)}$; $s_\infty=\frac{1}{2}$.

$$\frac{1}{1.5}-\frac{1}{3.7}+\frac{1}{5.9}-\frac{1}{7.11}+, \&c.$$

Ans. $s_n=\frac{1}{6}\pm\frac{1}{2(2n+1)(2n+3)}$; $s_\infty=\frac{1}{6}$.

(17.) Find two numbers whose product $=45$; and the difference of their squares : the square of their difference : : 7 : 2. Ans. 9, and 5.

(18.) Prove the rules for Addition and Subtraction. (See § 14., 15., 16., and 17.)

(19.) Prove that in Multiplication and Division *like* signs give +, and *unlike* signs give —. (§ 18.)

(20.) Prove the rule for finding the greatest common measure of two quantities. (§ 30.)

(21.) Prove the rule for the transposition of terms in equations. (§ 40.)

(22.) Prove the rules for the extraction of the square and cube roots. (§ 46. and 47.)

(23.) Find the interest of P pounds for d days, at r per cent. Ans. $\frac{Pdr}{36500} = \frac{Pd}{73000}2r$.

(24.) One merchant owes another A pounds due in a days, B pounds due in b days, C pounds due in c days, &c.; find the equated time of payment. Ans. $\frac{Aa + Bb + Cc +, \&c.}{A + B + C +, \&c.}$.

(25.) Divide the number n into parts proportional to the numbers a, b, c, &c.

Ans. 1st part $= \frac{an}{a + b + c +, \&c.}$, 2d $= \frac{bn}{a + b + c +, \&c.}$, &c.

(26.) A number of merchants, M, N, P, &c., enter into partnership; M puts in A pounds for a days, N puts in B pounds for b days, P puts in C pounds for c days, &c.; they gain g pounds: what is the share of each?

Ans. M's share $= \frac{Aag}{Aa+Bb+Cc+, \&c.}$, N's $= \frac{Bbg}{Aa+Bb+Cc+, \&c.}$,

P's $= \frac{Ccg}{Aa + Bb + Cc +, \&c.}$, &c.

(27.) Find A, the amount of P pounds for n years, at r per cent. compound interest; denoting by a the amount of £1 for 1 year, at the same rate. Ans. $A = Pa^n$.

(28.) Find P, the present worth of A pounds due n years hence, at r per cent. compound interest.

Ans. $P = Aa_1{}^{-n}$.

(29.) In an equation of the form $ax = b$; suppose that when any number n is substituted for x, the result is $an = c$; prove that $c : n :: b : x$, and thence derive the common rule for single position.

(30.) In an equation of the form $ax + b = cx + d$, or $(a - c)x + b - d = 0$; suppose that when any two numbers m and n are substituted successively for x, the results are respectively

$$(a-c)m+b-d=e_1,$$

and $$(a-c)n+b-d=e_2;$$

prove that $e_1-e_2 : m-n :: e_1 : m-x$.

Also, $e_1-e_2 : m-n :: e_2 : n-x$

and $x=\frac{e_1 n - e_2 m}{e_1-e_2}$; thence derive two rules for double position.

(31.) Find A, the amount of an annuity of P pounds for n years, at r per cent. compound interest; denoting by a_1 the amount of £1 for 1 year, at the same rate.

Ans. $A = P(1+a_1+a_1{}^2+\ldots a_1{}^{n-1}) = P\frac{a_1{}^n-1}{a_1-1}$.

(32.) If A denote the amount of an annuity of P pounds for n years, at r per cent. compound interest, and a the amount of £1 for one year, at the same rate; prove that the present value $=\frac{A}{a^n}=\frac{1-a^{-n}}{a-1}P$.

(33.) If the annuity is to continue for ever, prove that its present value $=\frac{P}{a-1}$; or $r : P :: 100$: present value.

(34.) Apply the binomial theorem to obtain the 5th root of $\frac{1}{100002}$ to 20 places of decimals.

Ans. 999, 996, 000, 047, 999, 296, 011.

(35.) The perimeter of a right angled triangle is 24 feet, and the base is 8 feet; find the other two sides.

Ans. 6, and 10.

(36.) A besieged place, garrisoned by 10,000 men, was victualled for 27 days; but after 9 days 2500 men cut their way out: how long would the provisions last the survivors, supposing the daily rations to remain undiminished?

Ans. 24 days.

(37.) Prove that every fraction may be expressed either as a decimal with a finite number of places, or else as a recurring decimal.

(38.) Given $\frac{x}{2}+\frac{3x-8}{14}-\frac{4x+16}{11}=8$, to find x.

Ans. $x=28\frac{16}{27}$.

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(39.) Given $3x + 5y = 171$, $7x + 2z = 209$, and $7y + 2z = 90$; to find x, y, and z.
Ans. $x = 32$, $y = 15$, and $z = -7.5$.

(40.) Two gunners keep up a fire on a battery from two different guns; the first, who had spent 24 shots before the second opened fire, discharges 8 shots to 7 of those of his comrade, but uses in 4 rounds only the same quantity of powder as the other in 3; how many shots must the second fire before he has consumed as much powder as the first?
Ans. 126.

(41.) It is required to cut a piece equal to a solid foot from a plank $2\frac{1}{2}$ inches thick and 8 inches wide.
Ans. Length = 86·4 inches.

(42.) I make a journey of 3040 miles on horseback, on foot, and by water; $3\frac{1}{2}$ times as much is performed on land as by water, and $2\frac{1}{4}$ times as much on horseback as on foot. How far did I travel on foot? Ans. $675\frac{5}{9}$ miles.

(43.) Given $3x + 2y = 118$; $x + 5y = 191$; to find x and y. Ans. $x = 16$, $y = 35$.

(44.) Given $3\sqrt{(112 - 8x)} = 19 + \sqrt{(3x + 7)}$; to find x.
Ans. $x = 6$, and 11·8368.

(45.) Given $xy = 8$; $(3 - y)z = 12$; $(2 - x)(4 - z) = 4$; to find x, y, and z. Ans. $x = 1.6$, $y = 5$, $z = -6$.

(46.) The fore-wheel of a carriage makes six revolutions more than the hind-wheel in going 120 yards; but if the circumference of each wheel be increased 1 yard, the fore-wheel will make only 4 revolutions more than the hind-wheel in the same distance: required the circumference of each wheel. Ans. 4, and 5 yards.

(47.) If a cubic foot of metal weighs 4 cwts. 1 qr., and is worth ten guineas per ton; what will be the cost of a mile of piping made out of it with a 9-inch bore and $\frac{5}{8}$ of an inch thick? Ans. £903, 11s. $10\frac{1}{3}$d.

(48.) How many cubic feet of water are contained in a ditch shaped like the frustum of a wedge, 120 yards long, 6 feet deep, 10 yards broad at the top and 4 at the bottom.
Ans. 45360 cubic feet.

(49.) Three men set out at the same time from three stations in the same straight line, each distant 25 miles from

the next. They travel in straight lines at the rates of $2\frac{1}{2}$, 3, and $3\frac{1}{2}$ miles respectively per hour, and all arrive at the same time at the same point. How far has each travelled?

Ans. 125, 150, and 175 miles.

(50.) Two trains set out at the same time, one from London to Plymouth, and the other from Plymouth to London. The latter travelled 2 miles an hour faster than the former from Plymouth to Exeter, and 4 miles an hour faster from Exeter to Bristol. They arrived at the same instant at Bristol. At what rate did each travel; the distance from London to Bristol being 118 miles, from Bristol to Exeter 75 miles, and from Exeter to Plymouth 53 miles?

Ans. Train from London 37·1 miles per hour nearly.

THE END.